On the modal logic of weak filters and ultrafilters: tableaux, decidability, complexity, variants

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Abstract

The '*in many cases*' modality is used to handle patterns of default reasoning, like '*normally* φ *holds*' or '*probably* φ *is the case*'. It is of interest in Knowledge Representation and it has found interesting applications in *epistemic logic*, *typicality logics*, and '*normality*' conditionals. Such a generalized '*most*' quantifier can be interpreted over collections of '*large*' subsets of a set W of possible worlds. The notion of '*weak filter*' has been introduced independently by K. Schlechta and V. Jauregui as an incarnation of such a collection and *the modal logic of weak filters* has been axiomatized by V. Jauregui, providing a minimal logic of the '*in many cases*' modality. In this paper we contribute to the study of this modality, providing results on the '*majority logic*' $\mathbf{\Theta}$ of V. Jauregui. We provide a tableaux proof procedure for $\mathbf{\Theta}$ and prove its soundness and completeness with respect to the class of Scott-Montague semantics based on weak filters. The tableaux-based decision procedure allows us to prove that the satisfiability problem for $\mathbf{\Theta}$ is NP-complete. We discuss an alternative interpretation of '*majorities*' differentiating between finite and infinite sets of worlds and we prove that it comes at the high cost of destroying the finite model property for the resulting logic. Then, we show how to extend our results for the *modal logic of weak ultrafilters*, suited for applications where either a proposition or its negation (but not both) have to be considered '*true in many cases*', a notion useful in epistemic logic.

1 Introduction

Commonsense Reasoning often deals with patterns of *default reasoning* captured by sentences of the form '*in most days, Jim will have a coffee after work*' or '*in almost all cases, rain causes a terrible traffic jam in Athens*'. These are typical examples of an inference pattern which falls within a certain scheme: a fact is considered to be *plausibly true* if it holds in '*many*' ('*most*', '*almost all*', '*a majority of*') *states of affairs* the agent can think or imagine of; thus, it can be considered to be '*true by default*', in case there is no information to the contrary.

This is one, out of several forms of Commonsense Reasoning that involve 'weighting' of possible cases: *nonmonotonic reasoning* is interested in the most '*normal*' situations, *conditional logic* is interested in the worlds *closest* to the *actual* world, *deontic logic* is interested in the *morally acceptable* worlds and *plausibility logics* are interested in *'large' subsets* of possible worlds. The interpretation of '*normally*' as a '*most*' ('*in many cases*') modality - a generalized majority quantifier - is of interest in *Knowledge Representation* and, as a notion of qualified truth, can be naturally handled with the machinery of Modal Logic, in as much the same way as '*necessity*', '*knowledge*' and '*belief*'. The modality of interest is variably read as '*in many cases*', '*in almost all cases*', '*majority true*' [Jau08], '*probably true*' [Her03, Bur69], '*true by default*' [Jau07] and has been employed in *Epistemic Logic* in order to capture weak notions of belief [Her03, AKZ12, KMZ14]. Most certainly, the study of logics for this modality is of interest for KR.

Of course, the main issue is the **model-theoretic interpretation** of '*most*'. **Which subsets count as 'majorities' in a set of possible worlds?** The question is not at all new; it has been addressed early enough in classical *Model Theory*, in the endeavour to define **generalized quantifiers**: '*we can regard generalized quantifiers as operations which pick out certain subsets from among all the subsets of a domain of a given realization. Thus, the existential quantifier picks out the non-empty sets and the universal quantifier just the whole domain itself*' [BS69, p. 261]. The situation is similar in our context: a **normal modal** ('*necessity*') **operator** is a **bounded universal quantifier** and we wish to axiomatize a **'majority' modality**, conceived as a **generalized 'most' quantifier**. And actually, the notion of (collections of) '*large*' sets we use in this paper, originated in K. Schlechta's work on generalized quantifier semantics for defaults [Sch95]. The same notion of a **weak filter** was independently discovered by V. Jauregui [Jau08] who also axiomatized the modal logic of weak filters. The definitions for this and other notions of 'majority' can be found in Section 3.1 below.

In this paper we work on the logic $\mathbf{\Theta}$ of weak filters from [Jau08], a minimal logic of the '*in many cases*' modality, in the sense that weak filters capture a *minimum* set of requirements expected to hold in any definition of 'large' sets; see Remark 3.2 below. We provide a sound and complete tableaux-based proof procedure for $\mathbf{\Theta}$ in Section 4. The decision procedure allows us to pin down the complexity of $\mathbf{\Theta}$ which is shown to be at the lower level expected for a modal logic (Section 5.2), completing thus the picture for the computational properties of $\mathbf{\Theta}$.

We further contribute to the study of the '*majority*' modality by investigating in Section 7 the logic of **weak ultrafilters** which are maximal ('complete') weak filters corresponding to negation-complete 'normality' theories. We show that our results carry through in this case with a reasonable number of adjustments to the machinery we provide. En route, we discuss in Section 6 the possibility of maintaining a different notion of 'majority' for the finite and the infinite case, keeping the natural and meaningful '*clear majority*' case (exceeding half the cardinality of the 'universe') for the finite case. This certainly makes sense and it works ([Zik12]); yet, it comes at the cost of destroying the finite model property, as we prove with a combinatorial argument. Finally, in Section 8 we discuss related work and future research questions.

2 Background Material

Modal Logic traditionally **studies logics of qualified truth**: '*necessarily true*', '*known or believed to be true*', '*henceforth true*', are some of the most important and well-known interpretations of the modal operator, in areas such as *epistemic, doxastic* and *temporal logic*. We only provide some basic definitions below and we assume that the reader is acquainted with the notions and techniques of Modal Logic. For a complete treatment, the reader is referred to [HC96, Gol92, BdRV01]. For a readable and thorough treatment of modal tableaux, we refer the reader to the books of M. Fitting [Fit83, FM98], whose methods, notation and terminology we use in this paper.

The language of propositional modal logic, extends classical logic with a modal operator $\Box\varphi$, traditionally read as '*necessarily* φ'. In this paper, it will be read as '*in most cases* φ'. A dual '*possibly* φ' operator $\Diamond\varphi$ is just an abbreviation for $\neg\Box\neg\varphi$. *Modal* logics are sets of modal formulas containing classical propositional logic (i.e. containing all tautologies in the augmented language $\mathcal{L}_\Box$) and closed under rule

$$\textbf{MP}.\frac{\varphi, \varphi \supset \psi}{\psi}$$

The smallest modal logic is denoted as **PC** (propositional calculus in the augmented language). **Normal** are called those **modal logics**, which contain all instances of axiom **K**. $\Box\varphi \wedge \Box(\varphi \supset \psi) \supset \Box\psi$ and are closed under rule **MP**, Uniform Substitution and the rule

$$\textbf{RN}.\ \frac{\varphi}{\Box\varphi}$$

By $\mathbf{KA_1 \ldots A_n}$ we denote the normal modal logic axiomatized by axioms $\mathbf{A_1}$ to $\mathbf{A_n}$.

Relational possible-worlds models. Normal modal logics are interpreted over **Kripke** (or relational) possible-worlds models: a *Kripke model* $\mathfrak{M} = \langle W, \mathcal{R}, V\rangle$ consists of a set W of *possible worlds* (*states, situations*), a *binary accessibility relation* between them $\mathcal{R} \subseteq W \times W$ and a valuation $V : \Phi \to 2^W$ assigning to each propositional variable from Φ a set of worlds. V extends uniquely to a valuation $\overline{V}$ for all well-formed formulæ of the modal language. The pair $\mathfrak{F} = \langle W, \mathcal{R} \rangle$ is called a **Kripke** (*relational*) **frame**. The normal modal logics correspond exactly to the relational possible worlds models; to define weaker modal logics, one has to resort to other modal semantics.

Scott-Montague possible-worlds models. The so-called *Scott-Montague semantics*, also called **neighborhood semantics** or **minimal models** [Che80], were introduced independently by D. Scott and R. Montague. The reader is referred to [Che80] and [Seg71, Pac07] for details. In *Scott-Montague models*, each possible world w is associated to its '*neighborhood*': a collection of sets of possible worlds, intended to be a collection of '*propositions*' *necessary* at w. A **neighborhood model** is a triple $\mathfrak{N} = \langle W, \mathcal{N}, V \rangle$, where W is a set of possible worlds, $\mathcal{N} : W \to 2^{2^W}$ is a **neighborhood function** assigning to a possible world its '*neighborhood*' and V is again a valuation extending uniquely to a $\overline{V}$ over all formulæ of the language. Inside a state w, formulæ of the form $\Box\varphi$ become true at w iff the set of possible worlds $\overline{V}(\varphi)$ where φ holds (called the *truth set* of φ, also denoted as $|\varphi|$) $\overline{V}(\varphi) = \{v \in W \mid \mathfrak{N}, v \models \varphi\}$ (intuitively the 'proposition' denoted by φ inside $\mathfrak{N}$) is among the propositions populating the neighborhood of w: $|\varphi| \in \mathcal{N}(w)$. The pair $\mathfrak{F} = \langle W, \mathcal{N} \rangle$ is called a **Scott-Montague** (neighborhood) **frame**.

Using the machinery of Scott-Montague semantics, different families of modal logics can be defined. From the weaker to the strongest: *classical*, *monotonic*, *regular* and *normal modal logics* [Che80, Chapters 7 & 8]. Of interest to us in this paper, is the class of **monotonic modal logics** which are logics closed under the **rule of monotonicity**

$$\textbf{RM}.\ \frac{\varphi \supset \psi}{\Box\varphi \supset \Box\psi}$$

The smallest monotonic modal logic is determined by the class of Scott-Montague frames with **upwards-closed neighborhoods**, that is, frames in which each collection of possible worlds (neighborhood) is closed under the superset relation (also called **supplementations** in [Che80]).

3 Weak Filters and their modal logic

3.1 Weak Filters as collections of 'large' subsets

We assume a set W of '*states*' or '*possible worlds*'. **Which subsets of W would we accept to consider as large?** *Which subsets of W correspond to the phrase: 'many' states*? Well, '*many*' can be defined in a number of ways and there exist different real-life concepts of '*majority*' (*simple majority, overwhelming majority, significant majority, clear majority*, some of which is even difficult to define precisely). For a finite W, a rational answer could be to collect the subsets with cardinality strictly more than $|W|/2$, a *simple* (or *clear*) *majority*. This practically amounts to the first method for defining 'majorities' proposed by K. Schlechta in [Sch04]: *counting*. Another possibility could be via the *mathematical theory of measure*. The third one, which we use here, originated in *Set Theory* and *Model Theory*, and captures collections of 'large' subsets via *filters*.

A **filter over** a nonempty set W is a collection F of subsets of W, such that

- $W \in F$ and $\emptyset \notin F$
- $X \in F$ and $X \subseteq Y \subseteq W$ implies $Y \in F$
  (*filters are upwards closed*)
- $X \in F$ and $Y \in F$ implies $(X \cap Y) \in F$
  (*filters are closed under intersection*)

This definition disallows *the* improper filter over W, which is just the whole powerset algebra of W. **Ultrafilters** over W are the *maximal* (proper) *filters*, or equivalently, the filters satisfying in addition the following completeness requirement:

- for every $X \subseteq W$, either $X \in F$ or $(W \setminus X) \in F$

A filter of the form $F = \{Y \subseteq W \mid X \subseteq Y\}$, where X is a fixed non-empty subset of W, is called a **principal filter** over W, the principal filter **generated** by X. In case X is a singleton, F is a **principal ultrafilter**.

The following definition from [Jau08] introduces the notion of **weak filter**.

Definition 3.1 Let W be a non-empty set and $\mathcal{K} \subseteq 2^W$ be a non-empty collection of subsets of W. $\mathcal{K}$ is a **weak filter** over W iff it satisfies the following conditions:

(i) $W \in \mathcal{K}$ (*non-emptiness*)

(ii) $X \in \mathcal{K}$ and $X \subseteq Y \subseteq W$ implies $Y \in \mathcal{K}$ (*upwards closure*)

(iii) $X \in \mathcal{K}$ implies $(W \setminus X) \notin \mathcal{K}$ (*$\mathcal{K}$ cannot contain a set and its complement*)

In [Sch97] a different, but provably equivalent, notion of large sets had been given: it is essentially the same with the previous one, replacing the third condition for the following one:

(iv) If $X, Y \in \mathcal{K}$, then $X \cap Y \neq \varnothing$ (*pairwise coherence*)

Since (iii) and (iv) are provably equivalent (given (i) and (ii)), we will switch freely between them and use them under the same name. A notion of **weak ultrafilter** has been proposed in [AKZ12] by replacing item (iii) of Def. 3.1 with

(iii)′ $X \notin \mathcal{K} \;\Leftrightarrow\; (W \setminus X) \in \mathcal{K}$ (*exactly one, out of a set and its complement, is in* $\mathcal{K}$)

The class of **weak Ultrafilters** is of an independent set-theoretic interest. In [KMNZ15] various results are collected, including the fact that every weak filter can be extended to a weak ultrafilter, assuming the Axiom of Choice. Some comments are in order, with respect to the intuitive meaning of weak filters and ultrafilters.

Remark 3.2 The *weak filter* is a simple 'weakening' of a classical filter, obtained by relaxing the requirement of *closure under intersection* to the condition of *pairwise coherence*. Obviously every (ultra)filter is a weak (ultra)filter, but not vice versa [KMNZ15]. This raises some questions on the applicability of weak filters as embodiments of the notion of '*a collection of large subsets*', despite the fact that *genuinely* weak filters on finite and infinite sets come closer to this intuition than their classical counterparts. In any case, it is not reasonable to consider a *principal filter* or *ultrafilter* as a collection of 'large' subsets, given that they may contain definitely '*small*' sets, even singletons.

The reader should keep in mind that weak filters represent a mathematical abstraction, intended to capture only the minimum requirements, the essential ingredients of the intuitive abstraction of a 'large' subset. As stated by K. Schlechta '*a reasonable abstract notion of size without the properties of weak filters seems difficult to imagine. The full set seems to be the best candidate for a 'big' subset, 'big' should cooperate with inclusion, and finally no set should be 'big' and 'small' at the same time*' [Sch04].

3.1.1 Other notions of 'majority'

Other notions of collections of 'large sets' exist. A fine-grained definition of a collection F of 'majorities' has been given by E. Pacuit and S. Salame [PS04, Sal06]. Assuming a universe W, a collection F of subsets of W is a **majority space** iff

- either $X \in F$ or $(W \setminus X) \in F$
- $X \in F$, $Y \in F$ and $X \cap Y = \emptyset$ imply that $X = (W \setminus Y)$
- if X is a large set, and a finite subset of it is replaced by a set of greater cardinality that the one removed, a large set is obtained.

The motivation for this definition has to do with applications of graded modal logic [vdH92]. Table 1 summarizes the various approaches mentioned in this paper, so that the reader can grasp their similarities and differences.

	filter	*ultrafilter*	*weak filter* [Sch97, Jau08]	*majority space* [PS04, Sal06]	*weak ultrafilter* [AKZ12]
$C \neq \varnothing$	•		•		
$(\forall X \in C)(\forall Y \subseteq W)$ $(X \subseteq Y \Rightarrow Y \in C)$	•	•	•		•
$(\forall X, Y \in C)$ $X \cap Y \in C$	•	•			
$(\forall X \subseteq W)$ $(X \in C \Rightarrow W \setminus X \notin C)$		•	•		•
$(\forall X \subseteq W)$ $(X \notin C \Rightarrow W \setminus X \in C)$		•		•	•
$(\forall X, Y \in C)$ $(Y \neq W \setminus X \Rightarrow X \cap Y \neq \varnothing)$				•	
$(\forall X \in C)(\forall \text{ finite} F \subseteq X)$ $(\forall Y \subseteq W)$ $(X \cap Y = \varnothing \ \& \ F \leq_c Y \Rightarrow (X \setminus F) \cup Y \in C)$				•	

Table 1: Notions of '*collections of large subsets*'

3.2 Θ: the modal logic of weak filters

The modal logic of weak filters has been axiomatized in [Jau08]. We wish to pin down the principles governing the '*in many cases*' modality, when 'majority' is interpreted over weak filters. The resulting logic is not expected to be normal, as the axiom **K** does not cooperate smoothly with the 'majority' modality; see [Jau08] for a counterexample. Thus, we aim in axiomatizing the modal logic of the class of Scott-Montague frames, in which each neighborhood is a collection of '*large subsets*' of possible worlds, i.e. a weak filter over W.

Definition 3.3 (Θ-frame,[Jau08]) *A Scott-Montague frame* $\mathcal{F} = \langle W, \mathcal{N} \rangle$ *is a* **Θ-frame** *if for every* $w \in W$*,* $\mathcal{N}(w)$ *is a weak filter over* W*:*

1. $W \in \mathcal{N}(w)$
2. $\mathcal{N}(w)$ *is upwards closed*
3. $X \in \mathcal{N}(w)$ *implies* $W \setminus X \notin \mathcal{N}(w)$

The modal logic of the class of $\mathbf{\Theta}$-frames turns out to be a simple monotonic modal logic.

Definition 3.4 ([Jau08]) $\mathbf{\Theta}$ *is the smallest modal logic which contains the axioms*

$$\mathbf{N}.\ \Box\top$$

$$\mathbf{D}.\ \Box\varphi \supset \Diamond\varphi$$

and is closed under the rule

$$\mathbf{RM}.\ \frac{\varphi \supset \psi}{\Box\varphi \supset \Box\psi}$$

For those acquainted with the modal logics determined by classes of Scott-Montague frames, it is not hard to check that this is the logic of $\mathbf{\Theta}$-frames. The axiom **N** corresponds to the non-emptiness condition ($W \in \mathcal{N}(w)$, $\mathcal{N}(w)$ *contains the unit* in the terminology of [Che80]), the rule **RM** corresponds to *upwards closure* ($\mathbf{\Theta}$-frames are *supplemented* in the terminology of [Che80]) and the axiom **D** ensures the pairwise coherence of propositions inside each neighborhood [Che80, pp. 223-224]. The proofs of soundness and completeness of $\mathbf{\Theta}$ with respect to the class of $\mathbf{\Theta}$-frames can be found in [Jau08, Chapter 3]; they are typical canonical model arguments.

4 Tableaux for $\mathbf{\Theta}$

In this section we present a tableau system for $\mathbf{\Theta}$. We assume the reader has a working knowledge of tableaux proof procedures; we follow [Fit83] to which we refer for details. This logic, not involving axiom **B**, or a notion of symmetry in terms of Scott-Montague frames [Che80], suggests that a tableau system like the one used in [Fit83] for the logics $\mathbf{K}, \mathbf{T}, \mathbf{D}$ etc., can be adopted. Such a system indeed works; however, in order to develop a **systematic procedure** for finding (or not finding) a proof, we opt for a **prefixed tableau** system. Although we will not be using the prefixes to ultimately represent a notion of accessibility (there is none), the prefixes still provide a notation for naming worlds. A systematic procedure will not only be useful for proving *completeness*, but will allow us to prove *decidability* and the *finite model property*.

Some **terminology** is in order: The version of *prefix* used is simply a positive integer (the prefixes used for universal Kripke frames). A **prefixed formula** is an expression of the form $n\ \varphi$, where n is a prefix and φ is a formula. A tableau branch is *closed* if it contains both $n\ \varphi$ and $n\ \neg\varphi$ for some prefix n and formula φ. A tableau is *closed* if all of its branches are closed. A tableau or branch is *open* if it is not closed. The terminology and the techniques we use come from M. Fitting's work [FM98, Fit83].

4.1 Tableaux Rules

For the alphabet of our tableaux, we assume $\Diamond\varphi$, $\varphi \supset \psi$, $\varphi \equiv \psi$ are abbreviations for $\neg\Box\neg\varphi$, $\neg\varphi \vee \psi$, $(\varphi \supset \psi) \wedge (\psi \supset \varphi)$ respectively, thus no corresponding rules have to be specified.

Definition 4.1 *A* **Θ-tableau** *for a formula φ is a tableau that starts with the prefixed formula $1\ \neg\varphi$ and is extended using any of the rules below.*

[Double negation rule] $\dfrac{n\ \neg\neg\varphi}{n\ \varphi}$

[Conjunctive rules] $\dfrac{n\ \varphi \wedge \psi}{\begin{array}{c} n\ \varphi \\ n\ \psi \end{array}}$ $\dfrac{n\ \neg(\varphi \vee \psi)}{\begin{array}{c} n\ \neg\varphi \\ n\ \neg\psi \end{array}}$

[Disjunctive rules] $\dfrac{n\ \varphi \vee \psi}{n\ \varphi \mid n\ \psi}$ $\dfrac{n\ \neg(\varphi \wedge \psi)}{n\ \neg\varphi \mid n\ \neg\psi}$

[**D**-rule] $\dfrac{n\ \Box\varphi}{n\ \neg\Box\neg\varphi}$

[π_1-rule] $\dfrac{n\ \neg\Box\psi}{m\ \neg\psi}$ for any prefix m new to the branch.

[π_2-rule] $\dfrac{\begin{array}{c} n\ \Box\varphi \\ n\ \neg\Box\psi \end{array}}{\begin{array}{c} m\ \varphi \\ m\ \neg\psi \end{array}}$ for any prefix m new to the branch.

The double negation, conjunctive and disjunctive rules, are standard for the propositional part of any modal logic. Regarding [**D**-rule], as its name suggests, it takes care of axiom **D**. Next, **Θ is a monotonic modal logic**. The appropriate rule, is that for any pair $\Box\varphi, \Diamond\psi$ there is a world such that φ, ψ hold ([π_2-rule], see [Fit83, Chapter 6.13] regarding the Logic **U** and its tableaux). Finally, the effect of axiom **N** is reflected by [π_1-rule], since introducing a new prefix due to a single $\Diamond$-formula implies that it is true for a reason, and not by default. Note that φ can be the same as ψ.

Definition 4.2 *A closed* **Θ***-tableau for a formula φ is a* **Θ-tableau proof** *for φ.*

Example 4.3 *We give a tableau proof for axiom* **M***. $\Box(p \wedge q) \supset (\Box p \wedge \Box q)$ which is a theorem of* **Θ***.*

1	$\neg(\neg\Box(p \wedge q) \vee (\Box p \wedge \Box q))$				*(1)*
1	$\neg\neg\Box(p \wedge q)$				*(2)*
1	$\neg(\Box p \wedge \Box q)$				*(3)*
1	$\Box(p \wedge q)$				*(4)*
1	$\neg\Box p$		*1*	$\neg\Box q$	*(5)*
2	$p \wedge q$		*2*	$p \wedge q$	*(6)*
2	$\neg p$		*2*	$\neg q$	*(7)*
2	p		*2*	p	*(8)*
2	q		*2*	q	*(9)*

Lines (2) and (3) are by a conjunctive rule. Line (4) is from (2) by double negation rule. Line (5) is from (3) by a disjunctive rule. Lines (6) and (7) are from (4) and (5) by [π_2-rule]. Lines (8) and (9) are from (6) by conjunction. Then the tableau is closed.

We proceed to define soundness and completeness of the teableau proof procedure.

4.2 Soundness

We need first to define what is a **satisfiable set of prefixed formulæ**.

Definition 4.4 *Suppose S is a set of prefixed formulæ. We say S is* **Θ-satisfiable** *if there is a* Θ-model $\langle W, \mathcal{N}, V \rangle$ *and a function* θ : *prefixes* $\to W$ *such that for any* $n\ \phi \in S$*, it holds that* $\theta(n) \models \phi$.

A branch is Θ-satisfiable *if the set of prefixed formulæ on it is* Θ-satisfiable. *A branch is* Θ-satisfiable *if the set of prefixed formulæ on it is* Θ-satisfiable. *We say that a tableau is* Θ-satisfiable *if some branch of it is* Θ-satisfiable.

Proposition 4.5 *A closed tableau is not Θ-satisfiable.*

Proof. Suppose a tableau was closed and satisfiable. This means that for some formula φ and prefix n, both $n\ \varphi$ and $n\ \neg\varphi$ occur on a tableau's branch. By Def. 4.4 there exists a model $\langle W, \mathcal{N}, V \rangle$ and a function θ such that $\theta(n) \models \varphi$ and $\theta(n) \models \neg\varphi$. A contradiction. ∎

Proposition 4.6 *Applying any of the rules to a Θ-satisfiable tableau, gives another Θ-satisfiable tableau.*

Proof. Suppose we have a satisfiable tableau and let $\mathcal{B}$ be the branch to which we apply any of the tableau rules. If there is another branch $\mathcal{B}'$ that is satisfiable, then the resulting tableau is trivially also satisfiable. So we assume the only satisfiable branch is $\mathcal{B}$.

[**Conjunctive rules**]: Suppose $n\ \varphi \wedge \psi$ occurs on $\mathcal{B}$. By applying the corresponding rule we add $n\ \varphi$ and $n\ \psi$ to the end of $\mathcal{B}$. Since $\mathcal{B}$ is satisfiable, Def. 4.4 provides a aforementioned model $\langle W, \mathcal{N}, V\rangle$ and function θ such that $\theta(n) \models \varphi \wedge \psi$. Consequently $\theta(n) \models \varphi$ and $\theta(n) \models \psi$, so the extended branch is also satisfiable, using the same model and function θ. The other conjunctive rule, as well as the double negation rule, are treated similarly.

[Disjunctive rules]: Suppose $n\ \varphi \vee \psi$ occurs on $\mathcal{B}$. By applying the corresponding rule we split the end of $\mathcal{B}$, adding $n\ \varphi$ to the left fork and $n\ \psi$ to the right. Since $\mathcal{B}$ is satisfiable there is a model $\langle W, \mathcal{N}, V\rangle$ and a function θ such that $\theta(n) \models \varphi \vee \psi$. Consequently $\theta(n) \models \varphi$ or $\theta(n) \models \psi$, and so at least one extension of $\mathcal{B}$ is satisfiable. The other disjunctive rule is treated similarly.

[D-rule]: Suppose $n\ \Box\varphi$ occurs on $\mathcal{B}$. By applying the corresponding rule we add $n\ \neg\Box\neg\varphi$ to the end of $\mathcal{B}$. Since $\mathcal{B}$ is satisfiable there is a model $\langle W, \mathcal{N}, V\rangle$ and a function θ such that $\theta(n) \models \Box\varphi$. We already know that the axiomatization includes axiom **D** so it is valid in any **Θ**-*model*. Consequently $\theta(n) \models \neg\Box\neg\varphi$.

[π_2-rule]: Suppose $n\ \Box\varphi$ and $n\ \neg\Box\psi$ occur on $\mathcal{B}$. By applying the corresponding rule we add $m\ \varphi$ and $m\ \neg\psi$ to the end of $\mathcal{B}$, where m is new to $\mathcal{B}$. Since $\mathcal{B}$ is satisfiable there is a model $\langle W, \mathcal{N}, V\rangle$ and a function θ such that $\theta(n) \models \Box\varphi$ and $\theta(n) \models \neg\Box\psi$. That is $|\varphi| \in \mathcal{N}(n)$ and $|\psi| \notin \mathcal{N}(n)$. For the sake of contradiction suppose $|\varphi| \cap |\neg\psi| = \varnothing$. By Def. 3.3 $|\varphi| \subseteq |\psi| \in \mathcal{N}(n)$ and hence the contradiction. So there is a world w such that $w \models \varphi, \neg\psi$. We need only define $\theta(m) = w$ and the extended branch is satisfiable.

[π_1-rule]: Similarly to [π_2-rule], we know $\Box\top$ is valid. ∎

We are ready to prove soundness of our procedure.

Theorem 4.7 **[Soundness]** *If φ is not* **Θ**-*valid, there is no* **Θ**-*tableau proof for φ.*

Proof. If φ is not **Θ**-*valid* , there is a **Θ**-*model* $\mathfrak{M}$ and a world w of $\mathfrak{M}$, such that $\mathfrak{M}, w \models \neg\varphi$. So the tableau with the prefixed formula $1\ \neg\varphi$ is satisfiable, using the model $\mathfrak{M}$ and defining $\theta(1) = w$. Now for the sake of contradiction suppose there is a **Θ**-tableau proof for φ, so by applying tableau rules we get a closed tableau. But due to Prop. 4.6, the resulting tableau will also be satisfiable, hence the contradiction due to Prop. 4.5. ∎

4.3 Completeness

Towards the proof of completeness, we will provide a **systematic procedure** of applying the tableaux rules, making sure everything that can be derived actually is. If the *systematic procedure* fails to produce a proof, then it will actually construct a **Θ**-*model* satisfying $\neg\varphi$, a *counter-model* witnessing non-validity.

4.3.1 Systematic procedure

Notation $\Diamond n$ and $\Box n$ for some prefix n are sets (intended to serve as **registries** so as to remember $\Diamond$-formulæ and $\Box$-formulæ that were found on a branch).

Stage 1: Write down $1\ \neg\varphi$. Also $\Diamond 1 = \Box 1 = \varnothing$.

After stage k we stop when tableau is closed or all **occurrences** of formulæ are *finished* (see below). Otherwise we proceed with stage $k+1$.

Stage $k+1$: Reading the formulæ starting with the leftmost branch and from top to bottom, we encounter the first *unfinished* occurrence of a prefixed formula F.

1. If F is $n\ \neg\neg\varphi, n\ \varphi \wedge \psi, n\ \neg(\varphi \vee \psi), n\ \varphi \vee \psi, n\ \neg(\varphi \wedge \psi), n\ \Box\varphi$ use the appropriate rule, for each open branch including F. That is, for the disjunctive case we split the end of each branch and for the rest of the cases we just add the appropriate formulæ at the end of the branch, provided they do not already occur.

2. If F is $n\ \Box\varphi$, we add φ to $\Box n$. For each open branch $\mathcal{B}$ that includes F and for each formula $\psi \in \Diamond n$, if there is no prefix m such that $\mathcal{B}$ includes $m\ \varphi$ and $m\ \psi$, we add $m\ \varphi$ and $m\ \psi$, where m is now the smallest positive integer new to $\mathcal{B}$.

3. If F is $n\ \neg\Box\varphi$, we add $\neg\varphi$ to $\Diamond n$. For each open branch $\mathcal{B}$ that includes F and for each formula $\psi \in \Box n$, if there is no prefix m such that $\mathcal{B}$ includes $m\ \neg\varphi$ and $m\ \psi$, we add $m\ \neg\varphi$ and $m\ \psi$, where m is the smallest positive integer new to $\mathcal{B}$. If $\Box n = \varnothing$ (we repeat the same without the use of $\Box$-formulæ) if there is no prefix m such that $\mathcal{B}$ includes $m\ \neg\varphi$ for some prefix m, we add $m\ \neg\varphi$, where m is the smallest positive integer new to $\mathcal{B}$.

F might not fall into one of the above cases (e.g. $n\ P$, P atomic) but then we just skip it. After the above we declare that occurrence of F **finished**.

4.3.2 Construction of a counter-model

Notation. Given a branch of a tableau we define $[\varphi] = \{n \mid n\ \varphi \text{ is on the branch}\}$. We remind that, given a model, $|\varphi| = \overline{V}(\varphi)$ is the **truth set** of φ.

Definition 4.8 *Let $\mathcal{T}$ be an open tableau generated by the systematic procedure and $\mathcal{B}$ an open branch of $\mathcal{T}$. We define a model $\mathfrak{M} = \langle W, \mathcal{N}, V\rangle$ as follows:*

- *W is the set of prefixes on $\mathcal{B}$.*
- *If $n\ P$ for P atomic, occurs on the branch then $n \models P$. Otherwise $n \models \neg P$.*
- *$\mathcal{N}(n) = \{S \subseteq W \mid \exists\varphi \in \mathcal{L}_\Box$ such that $S \supseteq [\varphi]$ & $n\ \Box\varphi$ occurs on $\mathcal{B}\} \cup \{W\}$.*

Proposition 4.9 *$\mathfrak{M}$ is a $\mathbf{\Theta}$-model.*

Proof. W is of course non-empty. It remains to show that each $\mathcal{N}(w)$ satisfies Definition 3.3. Indeed, all $\mathcal{N}(w)$ contain W and are closed under supersets by definition. Now suppose $A, B \in \mathcal{N}(w)$. It must be the case that $A \supseteq [\varphi], B \supseteq [\psi]$ and

$w\ \Box\varphi, w\ \Box\psi$ appear on $\mathcal{B}$, for some formulæ φ, ψ. Due to the systematic procedure, we have applied $[D-rule]$ and $w\ \neg\Box\neg\psi$ appears on the branch. We have also applied [π-rule] so $u\ \varphi, u\ \neg\neg\psi$, and due to [Double negation rule] $u\ \psi$, also occur. That is $[\varphi]\cap[\psi]\neq\varnothing \Rightarrow A\cap B\neq\varnothing$. ∎

Proposition 4.10 [Key fact] *Let $\mathfrak{M}$ be a model as in Def. 4.8. For any prefix n and formula φ:*

(i) if $n\ \varphi$ occurs on $\mathcal{B}$ then $\mathfrak{M}, n\models\varphi$.

(ii) if $n\ \neg\varphi$ occurs on $\mathcal{B}$ then $\mathfrak{M}, n\models\neg\varphi$.

PROOF. The proof is by induction on the complexity of φ. ∎

- Base case: φ is atomic.

  If $n\ P$ occurs on $\mathcal{B}$ then $n\models P$ by definition. If $n\ \neg P$ occurs on $\mathcal{B}$ then $n\ P$ does not occur, because $\mathcal{B}$ is *open*, and again by definition $n\models\neg P$.

- Induction step:

  - φ is $\neg\psi$. If $n\ \neg\psi$ occurs on $\mathcal{B}$, due to the induction hypothesis $n\models\neg\psi$. If $n\ \neg\neg\psi$ occurs, having followed the systematic procedure, $n\ \psi$ also occurs. Due to the induction hypothesis $n\models\psi$.
  - φ is $\psi\wedge\chi$. If $n\ \psi\wedge\chi$ occurs on $\mathcal{B}$, having followed the systematic procedure, $n\ \psi$ and $n\ \chi$ also occur. Due to the induction hypothesis $n\models\psi$,$n\models\chi$ so $n\models\psi\wedge\chi$. If $n\ \neg(\psi\wedge\chi)$ occurs on $\mathcal{B}$, having followed the systematic procedure, $n\ \neg\psi$ or $n\ \neg\chi$ occurs. Due to the induction hypothesis $n\models\neg\psi$ or $n\models\neg\chi$ so $n\models\neg(\psi\wedge\chi)$. The disjunctive case is treated similarly.
  - φ is $\Box\psi$. Due to the induction hypothesis $[\psi]\subseteq|\psi|$ and $[\neg\psi]\subseteq|\neg\psi|$. If $n\ \Box\psi$ occurs on $\mathcal{B}$ then by definition $[\psi]\subseteq|\psi|\in\mathcal{N}(n)$. Consequently $n\models\Box\psi$. If $n\ \neg\Box\psi$ occurs we need to show that $|\psi|\notin\mathcal{N}(n)$. For the sake of contradiction suppose the opposite. Then either $|\varphi|=W$ or $n\ \Box\chi$ occurs for some formula χ such that $[\chi]\subseteq|\psi|$. In the first case $|\neg\varphi|=\varnothing\Rightarrow[\neg\varphi]=\varnothing$ which is absurd since $m\ \neg\psi$ occurs for some prefix m due to the systematic procedure. In the latter case, again due to the systematic procedure, $m\ \chi, m\ \neg\psi$ occur. Hence $[\chi]\cap[\neg\psi]\neq\varnothing\Rightarrow|\psi|\cap|\neg\psi|\neq\varnothing$ which is again absurd. The possibility case $\neg\Box\psi$ is treated similarly.

Theorem 4.11 [Completeness] *If φ has no $\mathbf{\Theta}$-tableau proof, φ is not $\mathbf{\Theta}$-valid.*

PROOF. Since φ has no $\mathbf{\Theta}$-*tableau* proof, the tableau generated by following the systematic procedure has an open branch from which we construct a counter model. $1\ \neg\varphi$ occurs on the branch, and by the Key Fact (Prop. 4.10) $1\models\neg\varphi$. So $\neg\varphi$ is $\mathbf{\Theta}$-satisfiable i.e. φ is not $\mathbf{\Theta}$-valid. ∎

Example 4.12 *Using the methods described, we will show that the axiom*

$$\mathbf{C}.\ (\Box p \wedge \Box q) \supset \Box(p \wedge q)$$

is not a theorem of $\mathbf{\Theta}$. *The resulting tree is quite large. For illustration purposes, we only follow one of the branches that will stay open.*

1	$\neg(\neg(\Box p \wedge \Box q) \vee \Box(p \wedge q))$	(1)
1	$\neg\neg(\Box p \wedge \Box q)$	(2)
1	$\neg\Box(p \wedge q)$	(3)
1	$\Box p \wedge \Box q$	(4)
2	$\neg(p \wedge q)$	(5)
1	$\Box p$	(6)
1	$\Box q$	(7)
2	$\neg p$	(8)
1	$\neg\Box\neg p$	(9)
3	$\neg(p \wedge q)$	(10)
3	p	(11)
1	$\neg\Box\neg q$	(12)
4	$\neg(p \wedge q)$	(13)
4	q	(14)
5	$\neg\neg p$	(15)
5	p	(16)
6	$\neg\neg p$	(17)
6	q	(18)
3	$\neg q$	(19)
7	$\neg\neg q$	(20)
7	p	(21)
8	$\neg\neg q$	(22)
8	q	(23)
4	$\neg p$	(24)

After (3)	
$\Diamond 1$	$\Box 1$
$\neg(p \wedge q)$	

After (7)	
$\Diamond 1$	$\Box 1$
$\neg(p \wedge q)$	p
	q

After (9)	
$\Diamond 1$	$\Box 1$
$\neg(p \wedge q)$	p
$\neg\neg p$	q

After (12)	
$\Diamond 1$	$\Box 1$
$\neg(p \wedge q)$	p
$\neg\neg p$	q
$\neg\neg q$	

The counter-model $\mathcal{M} = \langle W, \mathcal{N}, V\rangle$ *derived from this branch has* $W = \{1, \ldots, 8\}$, $V(p) = \{3, 5, 6, 7\}, V(q) = \{4, 6, 7, 8\}$. *The only prefix with* $\Box$*-formulæ is* 1 *so for* $w \neq 1$ *we have* $\mathcal{N}(w) = W$ *and* $\mathcal{N}(1)$ *contains all supersets of* $|p|$ *and* $|q|$. *We refrain from writing them down explicitly.*

5 Decidability and Complexity

5.1 Finite model property

Notation $S(\varphi) = \{\chi \in \mathcal{L}_\Box \mid \chi \text{ is } \psi, \neg\psi, \neg\neg\psi \text{ or } \neg\Box\neg\psi, \text{ where } \psi \text{ a subformula of } \varphi\}$.

The following easy facts will prove to be useful.

Fact 5.1 *All formulæ occurring on a* $\mathbf{\Theta}$*-tableau for a formula* φ *belong to* $S(\varphi)$*. An easy proof is by induction on the number of rules applied.*

Fact 5.2 $S(\varphi)$ *is finite, in fact of size* $O(m)$*.*

Fact 5.3 *A* $\mathbf{\Theta}$*-tableau branch has a finite amount of prefixes, in fact* $O(m^2)$*.*

For a prefix to be introduced, a $\neg\Box$-formula or a combination of one $\neg\Box$-formula with a $\Box$-formula (with the same prefix) is needed on the branch.

Let φ be a $\mathbf{\Theta}$-satisfiable formula. Then an attempt to prove $\neg\varphi$ using the systematic procedure will fail. So there exists an open branch from which we can construct a model (Def. 4.8) that satisfies φ. The number of prefixes on the branch is finite (Fact 5.3), and so the model derived will also be finite.

5.2 Complexity

It has been shown that the satisfiability problem for a multi-agent extension of the smallest (epistemic) monotonic modal logic is in **NP** ([Var89], the logic $\mathcal{E}_{\{3\}}$). Logic $\mathbf{\Theta}$ deals with the reasoning of a single agent, and occurs from the smallest monotonic modal logic by adding axioms **N** and **D**. Assuming tableaux can provide the basis of an **NP** algorithm for the satisfiability in $\mathcal{E}_{\{3\}}$, one may guess that also $\mathbf{\Theta}$-SAT is in **NP**; the [**D**-rule] and [π_1-rule] rules reflect the effect of these two axioms on $\mathbf{\Theta}$-models and intuitively, they do not seem to burden the complexity. We prove this is indeed the case; the systematic procedure in Section 4.3.1 will be used in the **NP** algorithm mentioned.

Let a formula φ be the 'input' for the satisfiability problem, and size(φ) $= m$.

Algorithm $\mathbf{\Theta}$-sat

1. Run the systematic procedure for the tableau starting with $1\ \varphi$ with the exception that when you read a disjunction formula, non-deterministically choose which subformula to keep.
2. When the procedure stops
   (a) if it was because the branch closed, answer NO
   (b) else, answer YES.

Remark 5.4 *The resulting tableau has only one branch; using non-determinism we chose a single (computation) path.*

The *correctness* of the algorithm, based on our soundness and completeness results, is evident. However, we must show it is indeed an algorithm.

Proposition 5.5 *The algorithm described terminates in a finite number of steps.*

PROOF. Assume not. Then the single resulting branch is infinite. Due to Fact 5.2 and the fact that existing formulæ do not get added again, only a finite number of formulæ can occur with the same prefix. So it must be the case that there are infinite prefixes. Using Fact 5.3 we derive a contradiction. ∎

Proposition 5.6 **Θ**-*SAT is in* **NP**.

PROOF. Reading or writing any formula takes time $O(m)$. The algorithm described is non-deterministic, and for any choice made, it holds that each prefix has $O(m)$ formulæ and there are $O(m^2)$ prefixes. So (i) checking if a formula already exists,(ii) checking if the tableau is closed, (iii) adding prefixes for each pair of $\Box$ and $\neg\Box$ formulæ, these all can be accomplished in polynomial time. ∎

Proposition 5.7 **Θ**-*SAT is* **NP-hard**.

PROOF. Satisfiability in propositional logic is a special case of **Θ**-SAT. ∎

Theorem 5.8 **Θ**-*SAT is* **NP-complete**.

6 A digression: 'majorities' on finite vs infinite sets of worlds

Let us make a digression here and return to the discussion on what really deserves to be called a 'large' subset of W, keeping in mind the definitions of Section 3.1 and in particular Remark 3.2. It is tempting to try to alter the definition of 'large' subsets, keeping the original definition of weak filters for the infinite case and allowing only 'clear' majorities for the finite case. That is, allowing only the subsets which exceed half the cardinality of W, when W is finite. This, definitely makes sense. It disallows the problematic cases of principal filters which may well include 'small' subsets and comes closer to a reasonable notion of 'majority'. We prove here that, despite its intuitive appeal, the resulting logic will not have the finite model property; a high cost.

Assume that we adopt the following definition for '*a collection of large subsets of* W':

Definition 6.1 *Let W be a non-empty set. A non-empty collection F of subsets of W is* **a collection of large subsets** *iff*

- *If $|W| > \omega$, same as in Definition 3.1.*
- *If $|W| = n$, $n \in \omega$, $B \in F \Rightarrow |B| > n/2$.*

Note that if F is a collection of large sets by Definition 6.1 it is also so by Definition 3.1. Collections of subsets of a finite W complying with the requirement of Definition 6.1 are *weak filters* indeed, and actually *weak ultrafilters* when n is odd (see [KMNZ15] for a proof).

We will now construct a modal formula which is majority-satisfiable, but not satisfiable in a '*clear majority*' finite model complying with Definition 6.1. To achieve this, we employ a simple combinatorial trick: we construct propositional formulæ which *share exactly one common satisfying assignment* to their propositional variables and thus *a possible world can simultaneously satisfy at most two of them.* And then, proceed to show that these formulæ cannot be simulaneously '*majority*'-true in a world, whose neighborhood should contain only truth sets exceeding half the cardinality of W, as Definition 6.1 requires.

Assume the propositional variables p, q, r, s and pick up the following ten (out of sixteen) propositional valuations:

Atoms \ Valuation	1	2	3	4	5	6	7	8	9	10
p	T	T	T	T	T	T	T	T	F	F
q	T	T	T	T	F	F	F	F	T	T
r	T	T	F	F	T	T	F	F	T	T
s	T	F	T	F	T	F	T	F	T	F

Employing a familiar technique from classical propositional logic, we construct the following propositional formulæ in disjunctive normal form, ensuring that they pairwise share exactly one common satisfying assignment, distinct for each pair.

$$\begin{aligned}
\varphi_1 \equiv\ & (p \wedge q \wedge r \wedge s) \vee (p \wedge q \wedge r \wedge \neg s) \vee (p \wedge q \wedge \neg r \wedge s) \vee (p \wedge q \wedge \neg r \wedge \neg s) \\
\varphi_2 \equiv\ & (p \wedge q \wedge r \wedge s) \vee (p \wedge \neg q \wedge r \wedge s) \vee (p \wedge \neg q \wedge r \wedge \neg s) \vee (p \wedge \neg q \wedge \neg r \wedge s) \\
\varphi_3 \equiv\ & (p \wedge q \wedge r \wedge \neg s) \vee (p \wedge \neg q \wedge r \wedge s) \vee (p \wedge \neg q \wedge \neg r \wedge \neg s) \vee (\neg p \wedge q \wedge r \wedge s) \\
\varphi_4 \equiv\ & (p \wedge q \wedge \neg r \wedge s) \vee (p \wedge \neg q \wedge r \wedge \neg s) \vee (p \wedge \neg q \wedge \neg r \wedge \neg s) \vee (\neg p \wedge q \wedge r \wedge \neg s) \\
\varphi_5 \equiv\ & (p \wedge q \wedge \neg r \wedge \neg s) \vee (p \wedge \neg q \wedge \neg r \wedge s) \vee (\neg p \wedge q \wedge r \wedge s) \vee (\neg p \wedge q \wedge r \wedge \neg s)
\end{aligned}$$

This is readily checked from the following table:

	Satisfied by
φ_1	valuations 1,2,3,4
φ_2	valuations 1,5,6,7
φ_3	valuations 2,5,8,9
φ_4	valuations 3,6,8,10
φ_5	valuations 4,7,9,10

Now, we are ready to prove the desired result.

Proposition 6.2 *The formula* $\psi = \Box\varphi_1 \wedge \Box\varphi_2 \wedge \Box\varphi_3 \wedge \Box\varphi_4 \wedge \Box\varphi_5$

(i) is satisfiable in an infinite $\mathbf{\Theta}$*-model.*

(ii) is not satisfiable in a finite model of Definition 6.1.

PROOF. **(i)** We construct a model, in which the worlds in W are propositional valuations and $V(w) = w$ for any $w \in W$. We start with the ten valuations of the table above and add infinite copies of each of them. Then, we define

$$\mathcal{N}(w) = \{S \subseteq W \mid |\varphi_i| \subseteq S \text{ for some } i = 1, \ldots 5\}$$

It is easy to check that $w \models \psi$.

(ii) Suppose ψ is satisfied in a model of size n, at some world $w \in W$. Note that $||\varphi_i||$ is the cardinality of the truth set $|\varphi_i|$. We remind that, for a formula $\Box\chi$ to be true in a world w, the truth set $|\chi|$ should belong to the neighborhood of w: $|\chi| \in \mathcal{N}(w)$ (see Section 2).

- The formula ψ is a conjunction, so each $\Box\varphi_i$ is true in w. By construction, each φ_i corresponds to a different 'large' set in $\mathcal{N}(w)$.
- Each φ_i should be true in more than half of the worlds, so $\sum ||\varphi_i|| > 5n/2$.
- A single valuation satisfies at most two of the given formulæ φ_i. Thus, a world can belong to at most two truth sets. Summing up the cardinalities of the truth sets, we cannot exceed twice the size of W: $\sum ||\varphi_i|| \leq 2n$.

Then it has to be the case that $5n/2 < 2n$, a contradiction. ∎

A short explanation of the combinatorial trick is in order. We wanted to come up with formulæ $\varphi_1, \ldots, \varphi_m$ that impose different valuations such that n worlds are not enough. Each pair of these formulæ must have *at least* one common satisfying valuation, so that their truth sets can qualify as large subsets. We make it so that they have *exactly* one common satisfying valuation, unique for each pair. Given this we can close under supersets. Which is the smallest number of formulæ that can serve for this purpose? On the one hand, it has to be that $\sum ||\varphi_i|| > mn/2$ (Definition 6.1). On the other hand each world will satisfy at most two formulæ so $\sum ||\varphi_i|| \leq 2n$. For these two to lead to a contradiction we want $mn/2 > 2n \Rightarrow m > 4$. We therefore need five formulæ. Also, we need four propositional variables involved, which is the least amount so that there are at least $\binom{5}{2} = 10$ valuations available.

7 Θ_c: the modal logic of weak ultrafilters

We proceed now to the modal logic of **weak ultrafilters**, which are '*complete*' weak filters, in the sense that *exactly one* of a subset $X \subseteq W$ and its complement $W \setminus X$ should be present in the ultrafilter. In [KMNZ15] it is proved that this is a genuine notion (in the sense that there exist weak ultrafilters which are not classical ultrafilters), they possess interesting set-theoretic properties, and moreover the following variant of the *classical ultrafilter theorem* holds.

Proposition 7.1 ([KMNZ15]) *Consider a weak filter F over a non-empty set W. Then, assuming the Axiom of Choice, there exists a weak ultrafilter U over W extending F.*

We are interested in axiomatizing the modal logic Θ_c of the subclass of $\mathbf{\Theta}$-frames, in which neighborhoods are weak ultrafilters. It is not hard to see that it suffices to add in the axiomatization of Definition 3.4 the modal axiom $\mathbf{D_c}$. $\Diamond\varphi \supset \Box\varphi$, which, in combination with $\mathbf{D}$. $\Box\varphi \supset \Diamond\varphi$ guarantees the 'completeness' of the weak filters in the neighborhood of a possible world. The soundness and completeness of this Hilbert-style axiomatization of Θ_c is easy to prove, along the lines of [Jau08]. We will not do so, we will rather proceed to outline the modifications needed in our tableaux machinery for adjusting it to the logic Θ_c. It turns out that with a reasonable number of additions and minor modifications, we can prove soundness and completeness and identify the complexity of Θ_c.

Tableaux rules

- propositional rules

$$[\mathbf{CD}\text{-rule}] \quad \frac{n\ \neg\Box\varphi}{n\ \Box\neg\varphi}$$

$$[\pi\text{-rule}] \quad \frac{\begin{array}{c} n\ \Box\varphi \\ n\ \Box\psi \end{array}}{\begin{array}{c} m\ \varphi \\ m\ \psi \end{array}} \quad \text{for any prefix } m \text{ new to the branch.}$$

We still have axiom $\mathbf{D}$ in the axiomatization, however, by choosing to turn $\Diamond$ into $\Box$ we have no need for [$\mathbf{D}$-rule]. For the same reason [π_1-rule] and [π_2-rule] also become obsolete; we use the new [π-rule] instead. We do need a rule for a single $\Box$-formula; it is a matter of notation, φ can be the same as ψ.

Soundness [**CD**-rule]: We cannot of course rely on existing axiomatization and must use the definition of our (new) models. We have $\theta(n) \models \neg\Box\varphi \Rightarrow |\varphi| \notin \mathcal{N}(\theta(n)) \Rightarrow |\neg\varphi| \in \mathcal{N}(\theta(n)) \Rightarrow \theta(n) \models \Box\neg\varphi$.

[π-rule]: We have $\theta(n) \models \Box\psi \Rightarrow |\psi| \in \mathcal{N}(\theta(n)) \Rightarrow |\neg\psi| \notin \mathcal{N}(\theta(n)) \Rightarrow \theta(n) \models \neg\Box\neg\psi$. Now the proof follows as in Proposition 4.6.

Completeness **Systematic procedure:** everything is the same as in Section 4.3.1, except that there is no use for $\Diamond n$ sets and:

1. If F is $n\ \neg\neg\varphi, n\ \varphi\wedge\psi, n\ \neg(\varphi\vee\psi), n\ \varphi\vee\psi, n\ \neg(\varphi\wedge\psi), n\ \neg\Box\neg\varphi$ use the appropriate rule, for each open branch including F. That is, for the disjunctive case we split the end of each branch and for the remaining cases we just add the appropriate formulæ at the end of the branch provided they do not already occur.

2. If F is $n\ \Box\varphi$, we add φ to $\Box n$. For each open branch $\mathcal{B}$ that includes F and for each formula $\psi \in \Box n$, if there is no prefix m such that $\mathcal{B}$ includes $m\ \varphi$ and $m\ \psi$, we add $m\ \varphi$ and $m\ \psi$, where m is the smallest positive integer new to $\mathcal{B}$.

Existence of Counter-model: as in Definition 4.8. However we have to make sure the counter-model complies with the requirements for a weak ultrafilter:

- $\mathcal{N}^{-}(n) = \{S \subseteq W \mid \exists\varphi \in \mathcal{L}_{\Box}$ such that $S \supseteq [\varphi]$ & $n\ \Box\varphi$ occurs on $\mathcal{B}\} \cup \{W\}$. Then we take $\mathcal{N}(n)$ to be any complete extension. Such an extension exists.

We temporarily use the term '*existence*' and not '*construction*' of a counter-model. The reason is that the existence of a counter-model is now based on Proposition 7.1 which in turn is based on an equivalent of the *Axiom of Choice*. After discussing decidability for $\mathbf{\Theta}_c$ one should be convinced that also in this case, the number of prefixes, and therefore, the number of the worlds in the counter-model, is finite. And so a counter-model can be constructed algorithmically.

The proofs for the respective needed propositions, that the counter-model is indeed a $\mathbf{\Theta}_c$-model and the [Key fact], follow along the same lines.

Decidability and Complexity It is easy to see that all relevant remarks regarding $\mathbf{\Theta}$ still hold for $\mathbf{\Theta}_c$, perhaps with some changes in $S(\varphi)$. That is, each prefix has at most finite $O(m)$ formulæ and there are at most $O(m^2)$ prefixes. We can readily deduce the presence of the finite model property and that $\mathbf{\Theta_c}$-SAT is **NP-complete**.

8 Related Work and Future Research

The modal logic of weak filters is from [Jau07]. Yet, similar '*most*' modalities have appeared earlier in the Modal Logic and the Commonsense Reasoning literature.

In [Her03], a '*probably true*' modality is axiomatized, in combination with a *belief operator*; it is interesting that the axiomatization of '*probably*' is essentially the logic $\mathbf{\Theta}$ of V. Jauregui. The *belief & probability possible-words models* of [Her03] are similar to the *majority frames* (**Θ**-frames here) of [Jau07]; however, Herzig's work emphasizes in applications of this framework in more complex logics of action. A. Herzig attributes the essential ideas of his '*probably true*' operator to the earlier work of J. Burgess [Bur69], although the latter work has been written in the late sixties and uses algebraic techniques for examining a logic that adjoins a '*probably true*' operator to the well-known **S5** modal logic. The idea of 'large' subsets has been independently introduced in [Sch97].

We should note at this point that the '*in many cases true*' modality studied here, is readily suited for modelling weak notions of belief or notions of (qualitative) probability in the setting of Epistemic Logic; recent work is reported in [KMZ14, AKZ12]. For applications in *default reasoning* however, it has been argued that a '*normality*' modality does not suffice and the focus should be (and really is) on '*normality*' conditionals . The archetypical example in Non-Monotonic Reasoning is to infer that Tweety, a penguin, does not fly, although it is a bird. Assuming $\Box\varphi$ is a normality modality, representing the assertion '*birds typically fly*' as $\Box(bird \supset fly)$ or $(bird \supset \Box fly)$ is subject to criticism; the former falls prey to the '*paradoxes of strict implications*' [HC96] and the latter has been criticised within the KR community for carrying the same limitations as *circumscriptive* or *autoepistemic defaults* [Bou94]. Still, it remains interesting to study the '*in many cases*' modality, as it remains useful for *epistemic applications*, *typicality logics* and provides a foundation for proceeding to '*normality by majority*' conditionals for defeasible reasoning [Jau08, Chapter 4].

As for future work, the most interesting question is to find more accurate (and probably more complex) definitions of '*largeness*', investigate the emerging logics and compare the results to **Θ**, both in terms of expressiveness and their computational properties. It would be desirable to pin down the logic of '*clear majority*' in finite sets of possible worlds but this seems elusive; something at least as complex as the graded modal logic [vdH92] should be needed there.

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