A note on the paper titled "Composing Cardinal Direction Relations" [SK04]

Spiros Skiadopoulos
spiros@uop.gr

Manolis Koubarakis
koubarak@di.uoa.gr

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In an earlier work [SK04], we have presented a composition method for cardinal direction relations [Goy00, SGS+05, SK05, SK04]. Given two relations R_1 and R_2, the aforementioned method outputs the composition $R_1 \circ R_2$ in a form of a disjunction that contains all possible relations. The proposed method was applied to (a) the relations of set $\mathcal{D}$ that are realizable for the connected regions of set REG and (b) the relations of set $\mathcal{D}^*$ that are realizable for the (connected and disconnected) regions of set REG^*. Since $REG \subset REG^*$ holds, we also have $\mathcal{D} \subset \mathcal{D}^*$.

Later on, Zhang et al. [ZLLY08, ZLLY09] while studying a related operator, they discovered a case where the composition method presented in [SK04] (when restricted to the relations of set $\mathcal{D}$) does not compute the *strictest* disjunction. In other words, the composition method of [SK04] always returns a disjunction that contains *all* the sound results but there are some (rare, as we will later see) cases that the returned disjunction also contains disjuncts (i.e., relations) that might not belong to the composition. Notice that these cases arise only for the relations of the more restricted set $\mathcal{D}$ (defined exclusively for the connected regions of REG); for the more general set of relations $\mathcal{D}^*$ (that is defined for the connected and disconnected regions of REG^*) the composition method always returns the strictest disjunction. In the rest of this work, unless specifically stated, we will consider relations from set $\mathcal{D}$.

Let us see an example.

Example 1 *Consider the composition of relations $S{:}SW{:}W{:}NW{:}N$ and $N{:}NE$. The composition method of [SK04] outputs a disjunction having the following 130 relations (grouped according to the number of their tiles):*

N,

$NW{:}N$, $N{:}NE$, $B{:}N$,

$W{:}NW{:}N$, $NW{:}N{:}NE$, $N{:}NE{:}E$, $B{:}N{:}NE$, $B{:}NW{:}N$, $B{:}W{:}N$, $B{:}N{:}E$, $B{:}S{:}N$,

$SW{:}W{:}NW{:}N$, $W{:}NW{:}N{:}NE$, $NW{:}N{:}NE{:}E$, $N{:}NE{:}E{:}SE$, $B{:}S{:}N{:}NE$, $B{:}S{:}SW{:}N$, $B{:}S{:}NW{:}N$, $B{:}S{:}N{:}SE$, $B{:}W{:}NW{:}N$, $B{:}N{:}NE{:}E$, $B{:}NW{:}N{:}NE$, $B{:}S{:}W{:}N$, $B{:}W{:}N{:}E$, $B{:}S{:}N{:}E$, $B{:}W{:}N{:}NE$, $B{:}N{:}E{:}SE$, $B{:}SW{:}W{:}N$, $B{:}NW{:}N{:}E$,

$S{:}SW{:}W{:}NW{:}N$, $SW{:}W{:}NW{:}N{:}NE$, $W{:}NW{:}N{:}NE{:}E$, $NW{:}N{:}NE{:}E{:}SE$, $S{:}N{:}NE{:}E{:}SE$, $B{:}SW{:}W{:}NW{:}N$, $B{:}NW{:}N{:}NE{:}E$, $B{:}W{:}NW{:}N{:}NE$, $B{:}N{:}NE{:}E{:}SE$, $B{:}S{:}SW{:}W{:}N$, $B{:}W{:}NW{:}N{:}E$, $B{:}S{:}N{:}NE{:}E$, $B{:}S{:}W{:}NW{:}N$, $B{:}W{:}N{:}NE{:}E$, $B{:}S{:}N{:}E{:}SE$, $B{:}S{:}SW{:}NW{:}N$, $B{:}S{:}N{:}NE{:}SE$, $B{:}SW{:}W{:}N{:}NE$, $B{:}NW{:}N{:}E{:}SE$, $B{:}S{:}SW{:}N{:}NE$, $B{:}S{:}NW{:}N{:}SE$, $B{:}S{:}NW{:}N{:}NE$, $B{:}S{:}SW{:}N{:}SE$, $B{:}S{:}W{:}N{:}E$, $B{:}S{:}W{:}N{:}SE$, $B{:}SW{:}W{:}N{:}E$, $B{:}S{:}NW{:}N{:}E$, $B{:}S{:}W{:}N{:}NE$, $B{:}W{:}N{:}E{:}SE$, $B{:}S{:}SW{:}N{:}E$,

$S{:}SW{:}W{:}NW{:}N{:}NE$, $SW{:}W{:}NW{:}N{:}NE{:}E$, $W{:}NW{:}N{:}NE{:}E{:}SE$, $S{:}NW{:}N{:}NE{:}E{:}SE$, $S{:}SW{:}N{:}NE{:}E{:}SE$, $S{:}SW{:}W{:}NW{:}N{:}SE$, $B{:}S{:}SW{:}W{:}NW{:}N$, $B{:}W{:}NW{:}N{:}NE{:}E$, $B{:}S{:}N{:}NE{:}E{:}SE$, $B{:}SW{:}W{:}NW{:}N{:}NE$,

$B:NW:N:NE:E:SE$, $B:SW:W:NW:N:E$, $B:S:NW:N:NE:E$, $B:S:SW:W:N:SE$, $B:S:W:NW:N:NE$, $B:W:N:NE:E:SE$, $B:S:SW:N:E:SE$, $B:S:W:NW:N:E$, $B:S:W:N:NE:E$, $B:S:W:N:E:SE$, $B:S:SW:W:N:E$, $B:SW:W:N:NE:E$, $B:S:NW:N:E:SE$, $B:S:W:NW:N:SE$, $B:S:SW:W:N:NE$, $B:W:NW:N:E:SE$, $B:S:SW:N:NE:E$, $B:SW:W:N:E:SE$, $B:S:SW:NW:N:E$, $B:S:W:N:NE:SE$, $B:S:SW:NW:N:SE$, $B:S:NW:N:NE:SE$, $B:S:SW:NW:N:NE$, $B:S:SW:N:NE:SE$,

$S:SW:W:NW:N:NE:E$, $SW:W:NW:N:NE:E:SE$, $S:W:NW:N:NE:E:SE$, $S:SW:NW:N:NE:E:SE$, $S:SW:W:N:NE:E:SE$, $S:SW:W:NW:N:E:SE$, $S:SW:W:NW:N:NE:SE$, $B:SW:W:NW:N:NE:E$, $B:S:NW:N:NE:E:SE$, $B:S:SW:W:NW:N:SE$, $B:S:W:NW:N:NE:E$, $B:S:W:N:NE:E:SE$, $B:S:SW:W:N:E:SE$, $B:S:SW:W:NW:N:E$, $B:W:NW:N:NE:E:SE$, $B:S:SW:N:NE:E:SE$, $B:S:SW:W:NW:N:NE$, $B:S:SW:W:N:NE:SE$, $B:SW:W:NW:N:E:SE$, $B:S:SW:NW:N:NE:E$, $B:SW:W:N:NE:E:SE$, $B:S:SW:NW:N:E:SE$, $B:S:W:NW:N:NE:SE$, $B:S:SW:W:N:NE:E$, $B:S:W:NW:N:E:SE$, $B:S:SW:NW:N:NE:SE$,

$B:S:SW:W:N:NE:E:SE$, $B:S:SW:W:NW:N:E:SE$, $B:S:SW:W:NW:N:NE:SE$, $B:S:SW:W:NW:N:NE:E$, $B:SW:W:NW:N:NE:E:SE$, $B:S:W:NW:N:NE:E:SE$, $B:S:SW:NW:N:NE:E:SE$, $S:SW:W:NW:N:NE:E:SE$,

$B:S:SW:W:NW:N:NE:E:SE$.

From the above disjunction, the following 9 relations

$B:W:NW:NE:E$, $B:S:W:NW:NE:E$, $B:SW:W:NW:NE:E$, $B:W:NW:NE:E:SE$, $S:SW:W:NW:NE:E:SE$, $B:S:W:NW:NE:E:SE$, $B:S:SW:W:NW:NE:E$, $B:SW:W:NW:NE:E:SE$, $B:S:SW:W:NW:NE:E:SE$

are not realizable for connected regions. For instance, consider relation $B:W:NW:NE:E$*. Figure 1 illustrates regions* b *and* c *such that* $b\ N:NE\ c$ *holds. Using Figure 1, we can verify that there does not exists a connected region* a *satisfying both* $a\ B:W:NW:NE:E\ c$ *and* $a\ S:SW:W:NW:N\ b$*. Thus, relation* $B:W:NW:NE:E$ *does not belong to the composition of* $S:SW:W:NW:N$ *and* $N:NE$*.*

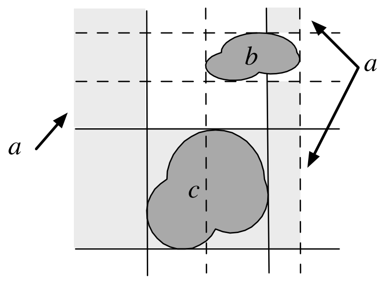

Figure 1: $B:W:NW:NE:E \notin S:SW:W:NW:N \circ N:NE$

$S:SW:W:NW:N$, $S:SW:W:NW:N$, $W:NW:N:NE:E$, $S:N:NE:E:SE$, $S:SW:W:E:SE$, $B:S:SW:NW:N$, $B:W:NW:NE:E$, $B:S:N:NE:SE$, $B:SW:W:E:SE$
$S:SW:W:NW:N:NE$, $S:SW:W:NW:N:NE$, $SW:W:NW:N:NE:E$, $W:NW:N:NE:E:SE$, $S:NW:N:NE:E:SE$, $S:SW:N:NE:E:SE$, $S:SW:W:NE:E:SE$, $S:SW:W:NW:E:SE$, $S:SW:W:NW:N:SE$
$B:SW:W:N:E:SE$, $B:S:SW:NW:N:E$, $B:S:W:NW:NE:E$, $B:S:W:N:NE:SE$, $B:SW:W:NE:E:SE$, $B:S:SW:NW:N:SE$, $B:SW:W:NW:NE:E$, $B:S:NW:N:NE:SE$, $B:SW:W:NW:E:SE$, $B:S:SW:NW:N:NE$, $B:W:NW:NE:E:SE$, $B:S:SW:N:NE:SE$
$S:SW:W:NW:N:NE:E$, $S:W:NW:N:NE:E:SE$, $S:SW:W:N:NE:E:SE$, $S:SW:W:NW:N:E:SE$
$SW:W:NW:N:NE:E:SE$, $S:SW:NW:N:NE:E:SE$, $S:SW:W:NW:NE:E:SE$, $S:SW:W:NW:N:NE:SE$
$B:S:SW:W:N:NE:SE$, $B:SW:W:NW:N:E:SE$, $B:S:SW:NW:N:NE:E$, $B:S:W:NW:NE:E:SE$, $B:SW:W:N:NE:E:SE$, $B:S:SW:NW:N:E:SE$, $B:S:SW:W:NW:NE:E$, $B:S:W:NW:N:NE:SE$
$B:SW:W:NW:NE:E:SE$ $B:S:SW:NW:N:NE:SE$
$B:S:SW:W:NW:NE:E:SE$ $B:S:SW:W:NW:N:NE:SE$, $B:SW:W:NW:N:NE:E:SE$, $B:S:SW:NW:N:NE:E:SE$
$S:SW:W:NW:N:NE:E:SE$

Table 1: Composition relations that need consideration

Let us assume that we want to compute the composition $R_1 \circ R_2$ of relations R_1 and R_2. Table 1 illustrates all cases of R_1 for which the composition method of [SK04] might not output the strictest relation. Each one of the 9 rows of Table 1 contains symmetric relations. Thus, we may only consider the first relation from each row of Table 1 (all other cases may be handled symmetrically), i.e., we have to consider the composition of just 9 cases.

Let us first consider the composition of relation $S{:}SW{:}W{:}NW{:}N$ (i.e., the first relation of the first row of Table 1) with an arbitrary relation R_2. [SK04] computes the composition $S{:}SW{:}W{:}NW{:}N \circ R_2$ using the formula:

$$\begin{array}{l}\{Q \in \mathcal{D}:\ (\exists s_1,\ldots,s_5)\ (Q = tu(s_1,\ldots,s_5) \wedge \\ \quad s_1 \in S \circ^* R_2 \wedge s_2 \in SW \circ^* R_2 \wedge s_3 \in W \circ^* R_2 \wedge s_4 \in NW \circ^* R_2 \wedge s_5 \in N \circ^* R_2\}\end{array} \tag{1}$$

where tu is the tile-union and operator $\circ^*$ can be computed by the composition operator $\circ$ by replacing all occurrences of δ with δ^*. Notice that although the implementation and the results of the composition method of [SK04] utilize Expression 1 and the $\circ^*$ operator, the actual text has a typo and refers to the $\circ$ operator.

Expression 1 guarantees that for any relation Q in the result set of Expression 1, the constraint set

$$S = \{\ a\ Q\ c,\ a\ S{:}SW{:}W{:}NW{:}N\ b,\ b\ R_2\ c\ \}.$$

can be satisfied by assigning connected regions to regions variables b and c. Unfortunately, as Example 1 indicates, Expression 1 cannot guarantee that there exist a solution of S that also assigns a connected region to region variable a.

To handle this situation, we start by considering a solution of S that assigns the connected regions α, β and γ to region variables a, b and c respectively and study the properties of region α. Our goal will be to define a predicate $Connected(Q)$ that evaluates to *true* if set S is satisfied by connected regions

We first note that if α is connected then $Q \in \mathcal{D}$. This check is performed by Expression 1 and it is not enough (see Example 1). Let us now consider regions $\alpha_1,\ldots,\alpha_k$ where $1 < k \leq 5$ (5 is the number of tiles in relation $S{:}SW{:}W{:}NW{:}N$) such that for every i, $1 \leq i \leq k$:

- α_i is a connected subregion of α (i.e., $\alpha_i \subset \alpha$).
- $\alpha_i\ P_1^i{:}\cdots{:}P_{t_i}^i\ \beta$ holds, where $1 \leq t_i < 5$, $\{P_1^i,\ldots,P_{t_i}^i\} \subset \{S, SW, W, NW, N\}$ and $P_1^i{:}\cdots{:}P_{t_i}^i \in \mathcal{D}$.

By the construction of regions $\alpha_1,\ldots,\alpha_k$, we also have:

$$\alpha_i\ tu(\sigma_1^i,\ldots,\sigma_{t_i}^i)\ \gamma$$

where $\sigma_j^i \in P_j^i \circ^* R_2$, $1 \leq j \leq t_i$ and $tu(\sigma_1^i,\ldots,\sigma_{t_i}^i) \in \mathcal{D}$.

Summarizing, if α is connected there will be some connected partitions

$$P_1^1{:}\cdots{:}P_{t_1}^1, \quad \ldots, \quad P_1^k{:}\cdots{:}P_{t_k}^k$$

of relation $S{:}SW{:}W{:}NW{:}N$ such that

$$tu(\sigma_1^1,\ldots,\sigma_{t_1}^1) \in \mathcal{D} \wedge \cdots \wedge tu(\sigma_1^k,\ldots,\sigma_{t_k}^k) \in \mathcal{D} \tag{2}$$

holds, where $\sigma_j^i \in P_j^i \circ^* R_2$, $1 \leq i \leq k$ and $1 \leq j \leq t_i$. For different values of $t_1,\ldots,t_k$ and k, Expression 2 we will give a different conjunction. Now, let us form predicate $Connected(Q)$

as the disjunction of all possible conjunctions that result in by Expression 2. We can verify that if α is connected then $Connected(Q)$ is true.

Predicate $Connected(Q)$ as previously defined can be very long and thus hard to evaluate. Luckily, by performing a case by case analysis we may significantly reduce the number of disjunctions that need consideration and, thus, define $Connected(Q)$ as follows:

$$(tu(s_1, s_2, s_3, s_4) \in \mathcal{D} \wedge tu(s_3, s_4, s_5) \in \mathcal{D}) \bigvee (tu(s_1, s_2, s_3) \in \mathcal{D} \wedge tu(s_2, s_3, s_4, s_5) \in \mathcal{D})$$

where $s_1 \in S \circ^* R_2$, $s_2 \in SW \circ^* R_2$, $s_3 \in W \circ^* R_2$, $s_4 \in NW \circ^* R_2$, $s_5 \in N \circ^* R_2$.

Similarly, we can define and simplify predicate $Connected(Q)$ for the remaining 8 cases. Table 2 illustrates the final result. Using $Connected(Q)$ we can restate Theorem 2 of [SK04] so that it will always result in the strictest disjunction.

Theorem 1 *Let $R_1 = R_{11}:\cdots:R_{1k}$ and R_2 be basic cardinal direction relations, where R_{11}, $\ldots$, R_{1k} are single-tile cardinal direction relations. Then*

$$R_1 \circ R_2 = \{Q \in \mathcal{D} : \; (\exists s_1, \ldots, s_k) \; (Q = tu(s_1, \ldots, s_k) \wedge Connected(Q) \wedge s_1 \in R_{11} \circ^* R_2 \wedge \cdots \wedge s_k \in R_{1k} \circ^* R_2)\}.$$

The proof of Theorem 1 is based on a case by case analysis. We have also verified our results with the consistency algorithm of Zhang et al. [ZLLY08, ZLLY09].

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	Relation R_1	Connected(Q)	where
1	$S:SW:W:NW:N$	$\big(tu(s_1,s_2,s_3,s_4)\in\mathcal{D}\wedge tu(s_3,s_4,s_5)\in\mathcal{D}\big)\bigvee$ $\big(tu(s_1,s_2,s_3)\in\mathcal{D}\wedge tu(s_2,s_3,s_4,s_5)\in\mathcal{D}\big)$	$s_1\in S\circ^* R_2, s_2\in SW\circ^* R_2, s_3\in W\circ^* R_2,$ $s_4\in NW\circ^* R_2, s_5\in N\circ^* R_2$
	Relations $S:SW:W:NW:N$, $W:NW:N:NE:E$, $S:N:NE:E:SE$, $S:SW:W:E:SE$, $B:S:SW:NW:N$, $B:W:NW:NE:E$, $B:S:N:NE:SE$ and $B:SW:W:E:SE$ are symmetric to Case 1 above.		
2	$S:SW:W:NW:N:NE$	$\big(tu(s_1,s_2,s_3,s_4)\in\mathcal{D}\wedge tu(s_3,s_4,s_5,s_6)\in\mathcal{D}\big)\bigvee$ $\big(tu(s_1,s_2,s_3)\in\mathcal{D}\wedge tu(s_2,s_3,s_4,s_5,s_6)\in\mathcal{D}\big)$	$s_1\in S\circ^* R_2, s_2\in SW\circ^* R_2, s_3\in W\circ^* R_2,$ $s_4\in NW\circ^* R_2, s_5\in N\circ^* R_2, s_6\in NE\circ^* R_2$
	Relations $S:SW:W:NW:N:NE$, $SW:W:NW:N:NE:E$, $W:NW:N:NE:E:SE$, $S:NW:N:NE:E:SE$, $S:SW:N:NE:E:SE$, $S:SW:W:NE:E:SE$, $S:SW:W:NW:E:SE$, $S:SW:W:NW:N:SE$ are symmetric to Case 2 above.		
3	$B:SW:W:N:E:SE$	$\big(tu(s_1,s_2,s_3,s_4,s_6)\in\mathcal{D}\wedge tu(s_3,s_4,s_5)\in\mathcal{D}\big)\bigvee$ $\big(tu(s_1,s_2,s_3)\in\mathcal{D}\wedge tu(s_2,s_3,s_4,s_5,s_6)\in\mathcal{D}\big)$	$s_1\in SW\circ^* R_2, s_2\in W\circ^* R_2, s_3\in B\circ^* R_2,$ $s_4\in E\circ^* R_2, s_5\in SE\circ^* R_2, s_6\in N\circ^* R_2$
	Relations $B:S:SW:NW:N:E$, $B:S:W:NW:NE:E$, $B:S:W:N:NE:SE$, $B:SW:W:NE:E:SE$, $B:S:SW:NW:N:SE$, $B:SW:W:NW:NE:E$, $B:S:NW:N:NE:SE$, $B:SW:W:NW:E:SE$, $B:S:SW:NW:N:NE$, $B:W:NW:NE:E:SE$ and $B:S:SW:N:NE:SE$ are symmetric to Case 3 above.		
4	$S:SW:W:NW:N:NE:E$	$\big(tu(s_1,s_2,s_3,s_4)\in\mathcal{D}\wedge tu(s_2,s_3,s_4,s_5)\in\mathcal{D}\wedge tu(s_4,s_5,s_6,s_7)\in\mathcal{D}\big)\bigvee$ $\big(tu(s_1,s_2,s_3,s_4)\in\mathcal{D}\wedge tu(s_3,s_4,s_5,s_6)\in\mathcal{D}\wedge tu(s_4,s_5,s_6,s_7)\in\mathcal{D}\big)\bigvee$ $\big(tu(s_1,s_2,s_3)\in\mathcal{D}\wedge tu(s_2,s_3,s_4,s_5,s_6)\in\mathcal{D}\wedge tu(s_4,s_5,s_6,s_7)\in\mathcal{D}\big)\bigvee$ $\big(tu(s_1,s_2,s_3,s_4)\in\mathcal{D}\wedge tu(s_2,s_3,s_4,s_5,s_6)\in\mathcal{D}\wedge tu(s_5,s_6,s_7)\in\mathcal{D}\big)$	$s_1\in S\circ^* R_2, s_2\in SW\circ^* R_2, s_3\in W\circ^* R_2,$ $s_4\in NW\circ^* R_2, s_5\in N\circ^* R_2, s_6\in NE\circ^* R_2,$ $s_7\in E\circ^* R_2$
	Relations $S:W:NW:N:NE:E:SE$, $S:SW:W:N:NE:E:SE$ and $S:SW:W:NW:N:E:SE$ are symmetric to Case 4 above.		
5	$SW:W:NW:N:NE:E:SE$	$\big(tu(s_1,s_2,s_3,s_4)\in\mathcal{D}\wedge tu(s_3,s_4,s_5,s_6,s_7)\in\mathcal{D}\big)\bigvee$ $\big(tu(s_1,s_2,s_3,s_4,s_5)\in\mathcal{D}\wedge tu(s_4,s_5,s_6,s_7)\in\mathcal{D}\big)$	$s_1\in SW\circ^* R_2, s_2\in W\circ^* R_2, s_3\in NW\circ^* R_2,$ $s_4\in N\circ^* R_2, s_5\in NE\circ^* R_2, s_6\in E\circ^* R_2,$ $s_7\in SE\circ^* R_2$
	Relations $S:SW:NW:N:NE:E:SE$, $S:SW:W:NW:NE:E:SE$, $S:SW:W:NW:N:NE:SE$ are symmetric to Case 5 above.		
6	$B:S:SW:W:N:NE:SE$	$\big(tu(s_1,s_2,s_3,s_4,s_6,s_7)\in\mathcal{D}\wedge tu(s_3,s_4,s_5)\in\mathcal{D}\big)\bigvee$ $\big(tu(s_1,s_2,s_3)\in\mathcal{D}\wedge tu(s_2,s_3,s_4,s_5,s_6)\in\mathcal{D}\big)$	$s_1\in SE\circ^* R_2, s_2\in S\circ^* R_2, s_3\in B\circ^* R_2,$ $s_4\in N\circ^* R_2, s_5\in NE\circ^* R_2, s_6\in SW\circ^* R_2,$ $s_7\in W\circ^* R_2$
	Relations $B:SW:W:NW:N:E:SE$, $B:S:SW:NW:N:NE:E$, $B:S:W:NW:NE:E:SE$, $B:SW:W:N:NE:E:SE$, $B:S:SW:NW:N:E:SE$, $B:S:SW:W:NW:NE:E$, $B:S:W:NW:N:NE:SE$ are symmetric to Case 6 above.		
7	$B:SW:W:NW:NE:E:SE$	$\big(tu(s_1,s_2,s_3,s_4)\in\mathcal{D}\wedge tu(s_2,s_3,s_4,s_7)\in\mathcal{D}\wedge tu(s_3,s_4,s_5,s_6)\in\mathcal{D}\big)\bigvee$ $\big(tu(s_1,s_2,s_3,s_7)\in\mathcal{D}\wedge tu(s_2,s_3,s_4,s_5)\in\mathcal{D}\wedge tu(s_2,s_3,s_4,s_6)\in\mathcal{D}\big)$	$s_1\in SW\circ^* R_2, s_2\in W\circ^* R_2, s_3\in B\circ^* R_2,$ $s_4\in E\circ^* R_2, s_5\in SE\circ^* R_2, s_6\in NE\circ^* R_2,$ $s_7\in NW\circ^* R_2$
	Relation $B:S:SW:NW:N:NE:SE$ is symmetric to Case 7 above.		
8	$B:S:SW:W:NW:NE:E:SE$	$\big(tu(s_1,s_2,s_3,s_4,s_6,s_7,s_8)\in\mathcal{D}\wedge tu(s_3,s_4,s_5,s_7,s_8)\in\mathcal{D}\big)\bigvee$ $\big(tu(s_1,s_2,s_3.s_6,s_7)\in\mathcal{D}\wedge tu(s_2,s_3,s_4,s_5,s_6,s_7,s_8)\in\mathcal{D}\big)$	$s_1\in NW\circ^* R_2, s_2\in W\circ^* R_2, s_3\in B\circ^* R_2,$ $s_4\in E\circ^* R_2, s_5\in NE\circ^* R_2, s_6\in SW\circ^* R_2,$ $s_7\in S\circ^* R_2, s_8\in SE\circ^* R_2$
	Relations $B:S:SW:W:NW:N:NE:SE$, $B:SW:W:NW:N:NE:E:SE$, $B:S:SW:NW:N:NE:E:SE$ are symmetric to Case 8 above.		
9	$S:SW:W:NW:N:NE:E:SE$	The union of all the results of Case 5 above	
9	all other relations	true	

Table 2: Defining predicate $Connected(Q)$