

## A NEW FAMILY OF MULTISTEP METHODS WITH IMPROVED PHASE-LAG CHARACTERISTICS FOR THE INTEGRATION OF ORBITAL PROBLEMS

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### ABSTRACT

In this paper, we introduce a new family of 10-step linear multistep methods for the integration of orbital problems. The new methods are constructed by adopting a new methodology which improves the phase-lag characteristics by vanishing both the phase-lag function and its first derivatives at a specific frequency. The efficiency of the new family of methods is proved via error analysis and numerical applications.

*Key words:* methods:  $N$ -body simulations – methods: numerical – solar system: general

### 1. INTRODUCTION

The numerical integration of systems of ordinary differential equations with oscillatory solutions has been the subject of research in the past decades. This type of ordinary differential equation (ODE) is often met in real problems, such as the  $N$ -body problem. For highly oscillatory problems, standard nonspecialized methods can require a huge number of steps to track the oscillations. One way to obtain a more efficient integration process is to construct numerical methods with an increased algebraic order, although the implementation of high algebraic order meets several difficulties (Quinlan 1999).

On the other hand, there are some special techniques for optimizing numerical methods. Trigonometrical fitting and phase fitting are some of them, producing methods with variable coefficients, which depend on  $v = \omega h$ , where  $\omega$  is the dominant frequency of the problem and  $h$  is the step length of integration. More precisely, the coefficients of a general linear method are found from the requirement that it integrates exactly powers up to degree  $p + 1$ . For problems with oscillatory solutions, more efficient methods are obtained when they are exact for every linear combination of functions from the reference set

$$\{1, x, \dots, x^K, e^{\pm\mu x}, \dots, x^P e^{\pm\mu x}\}. \quad (1)$$

This technique is known as exponential (or trigonometric if  $\mu = i\omega$ ) fitting and has a long history (Gautschi 1961; Lyche 1972). The set (1) is characterized by two integer parameters,  $K$  and  $P$ . The set in which there is no classical component is identified by  $K = -1$ , while the set in which there is no exponential fitting component (the classical case) is identified by  $P = -1$ . Parameter  $P$  will be called the level of tuning. An important property of exponential-fitted algorithms is that they tend to the classical ones when the involved frequencies tend to zero, a fact which allows us to say that exponential fitting represents a natural extension of the classical polynomial fitting. The examination of the convergence of exponentially fitted multistep methods is included in Lyche's theory (Lyche 1972). There is a large number of significant methods presented with high practical importance that have been presented in the bibliography (see, for example, Simos 2000, 2005, 2007;

Chawla & Rao 1986; Raptis & Allison 1978; Anastassi & Simos 2004, 2005a, 2005b, 2007; Lambert & Watson 1976; Cash & Mazzia 2006; Iavernaro et al. 2006; Mazzia et al. 2006; Berghe & Daele 2006; Psihoyios 2006). The general theory is presented in detail in Ixaru & Berghe (2004).

Considering the accuracy of a method when solving oscillatory problems, it is more appropriate to work with the phase lag, rather than the principal local truncation error (PLTE). We mention the pioneering paper of Brusa & Nigro (1980), in which the phase-lag property was introduced. This is actually another type of a truncation error, i.e., the angle between the analytical solution and the numerical solution. However, exponential fitting is accurate only when a good estimate of the dominant frequency of the solution is known in advance. This means that in practice, if a small change in the dominant frequency is introduced, the efficiency of the method can be dramatically altered. It is well known that for equations similar to the harmonic oscillator, the most efficient exponential-fitted methods are those with the highest tuning level.

In this paper, we present a new family of methods based on the 10-step linear multistep method of Quinlan & Tremaine (1990). The new methods are constructed by vanishing the phase-lag function and its first derivatives at a predefined frequency. Error analysis and numerical experiments show that the new methods exhibit improved characteristics concerning the long-term behavior of the solution in the five-outer problem. The paper is organized as follows. In Section 2, the general theory of the new methodology is presented. In Section 3, the new methods are described in detail. In Section 5, the stability properties of the new methods are investigated. Section 5 presents the results from the numerical experiments and finally, conclusions are drawn in Section 6.

### 2. PHASE-LAG ANALYSIS OF SYMMETRIC MULTISTEP METHODS

Consider the differential equations

$$\frac{d^2 y(t)}{dt^2} = f(t, y), \quad y(t_0) = y_0, \quad y'(t_0) = y'_0 \quad (2)$$

and the linear multistep methods

$$\sum_{j=0}^J a_j y_{n+j} = h^2 \sum_{j=0}^J b_j f_{n+j}, \quad (3)$$

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where  $y_{n+j} = y(t_0 + (n + j)h)$ ,  $f_{n+j} = f(t_0 + (n + j)h)$ ,  $y(t_0 + (n + j)h)$ , and  $h$  is the step size of the method. We associate the following functional to method (3):

$$L(h, a, b, y(t)) = \sum_{j=0}^J a_j y(t+j \cdot h) - h^2 \sum_{j=0}^J b_j y''(t+j \cdot h), \quad (4)$$

where  $a$  and  $b$  are the vectors of coefficients  $a_j$  and  $b_j$ , respectively, and  $y(t)$  is an arbitrary function. The algebraic order of the method (3) is  $p$ , if

$$L(h, a, b, y(t)) = C_{p+2} h^{p+2} y^{(p+2)}(t) + O(h^{p+3}). \quad (5)$$

The coefficients  $C_q$  are given as

$$\begin{aligned} C_0 &= \sum_{j=0}^J a_j \\ C_1 &= \sum_{j=0}^J j \cdot a_j \\ C_q &= \frac{1}{q!} \sum_{j=0}^J j^q \cdot a_j - \frac{1}{(q-2)!} \sum_{j=0}^J j^{q-2} b_j. \end{aligned} \quad (6)$$

The PLTE is the leading term of Equation (5):

$$\text{PLTE} = C_{p+2} h^{p+2} y^{(p+2)}(t). \quad (7)$$

The following assumptions will be considered in the rest of the paper:

1.  $a_j = 1$  since we can always divide the coefficients of Equation (3) with  $a_j$ .
2.  $|a_0| + |b_0| \neq 0$  since otherwise we can assume that  $J = J - 1$ .
3.  $\sum_{j=0}^J |b_j| \neq 0$  since otherwise the solution of Equation (3) would be independent of Equation (2).
4. The method (3) is at least of order one.
5. The method (3) is zero stable, which means that the roots of the polynomial

$$p(z) = \sum_{j=0}^J a_j z^j \quad (8)$$

all lie in the unit disk, and those that lie on the unit circle have multiplicity one.

6. The method (3) is symmetric, which means that

$$a_j = a_{J-j}, \quad b_j = b_{J-j}, \quad j = 0(1)J. \quad (9)$$

It is easily proved then that both the order of the method and the step number  $J$  are even numbers (Lambert & Watson 1976).

Consider now the test problem

$$y''(t) = -\omega^2 y(t) \quad (10)$$

where  $\omega$  is a constant. The numerical solution of Equation (10) by applying method (3) is described by the difference equation

$$\sum_{j=1}^{J/2} A_j(s^2)(y_{n+j} + y_{n-j}) + A_0(s^2)y_n = 0 \quad (11)$$

with

$$A_j(s^2) = a_{\frac{J}{2}-j} + s^2 \cdot b_{\frac{J}{2}-j} \quad (12)$$

and  $s = \omega h$ . The characteristic equation is then given as

$$\sum_{j=1}^{J/2} A_j(s^2)(z^j + z^{-j}) + A_0(s^2) = 0 \quad (13)$$

and the interval of periodicity  $(0, s_0^2)$  is then defined such that for  $s \in (0, s_0)$  the roots of Equation (13) are of the form

$$z_1 = e^{i\lambda(s)}, \quad z_2 = e^{-i\lambda(s)}, \quad |z_j| \leq 1, \quad 3 \leq j \leq J \quad (14)$$

where  $\lambda(s)$  is a real function of  $s$ . The phase lag (PL) of method (3) is then defined as

$$\text{PL} = s - \lambda(s) \quad (15)$$

and is of order  $q$  if

$$\text{PL} = c \cdot s^{q+2} + O(s^{q+4}). \quad (16)$$

In general, the coefficients of method (3) depend on some parameter  $v$ , thus the coefficients  $A_j$  are functions of both  $s^2$  and  $v$ . The following theorem has been proved by Simos & Williams (1999): for the symmetric method (10). The phase lag is given by

$$\text{PL}(s, v) = \frac{2 \sum_{j=1}^{J/2} A_j(s^2, v) \cdot \cos(j \cdot s) + A_0(s^2, v)}{2 \sum_{j=1}^{J/2} j^2 A_j(s^2, v)}. \quad (17)$$

We are now in position to describe the new methodology. In order to efficiently integrate oscillatory problems, a good technique is to calculate the coefficients of the numerical method by forcing the phase lag to be zero at a specific frequency. But, since the appropriate frequency is problem dependent and in general is not always known, we may assume that we have an error in the frequency estimation. It would be of great importance to force the phase lag to be less sensitive to this error. Thus, beyond the vanishing of the phase lag, we also force its first derivatives to be zero.

### 3. CONSTRUCTION OF THE NEW METHODS

#### 3.1. Base Method

The family of new methods is based on the 10-step linear multistep method of Quinlan & Tremaine (1990) which is of the form (3) with coefficients

$$\begin{aligned} a_0 &= 1, & a_1 &= -1, & a_2 &= 1, & a_3 &= -1, \\ & & a_4 &= 1, & a_5 &= -2, & & & \\ b_0 &= 0, & b_1 &= \frac{399187}{241920}, & b_2 &= -\frac{17327}{8640}, & b_3 &= \frac{597859}{60480}, \\ & & b_4 &= -\frac{704183}{60480}, & b_5 &= \frac{465133}{24192}. \end{aligned} \quad (18)$$

The principal term of the local truncation error (PLTE) of the method is given as

$$\text{PLTE} = \frac{52559}{912384} y^{(12)} h^{12}. \quad (19)$$

#### 3.2. Method PF-D0: Phase Fitted

The first method of the family (PF-D0) is constructed by forcing the phase-lag function to be zero at frequency  $V =$

$\omega \times h$ . Coefficients  $a$  are left the same, while coefficients  $b$  become

$$b_0 = 0, \quad b_1 = \frac{1}{8064} \frac{b_{1,num}^0}{D_0}, \quad b_2 = \frac{1}{4032} \frac{b_{2,num}^0}{D_0},$$

$$b_3 = \frac{1}{2016} \frac{b_{3,num}^0}{D_0}, \quad b_4 = \frac{1}{4032} \frac{b_{4,num}^0}{D_0}, \quad b_5 = \frac{1}{4032} \frac{b_{5,num}^0}{D_0},$$

where

$$D_0 = v^2((c)^4 + 6(c)^2 + 1 - 4c - 4(c)^3), \quad c = \cos(v), \quad s = \sin(v),$$

and

$$b_{1,num}^0 = -16128c^3 + 45139c^3v^2 + 6048c^2 - 73215c^2v^2 + 3024c + 47553cv^2 - 11917v^2 + 16128c^5 - 8064c^4 - 1008,$$

$$b_{2,num}^0 = -32256c^4 + 45139c^4v^2 - 64512c^3 + 24192c^2 - 22026c^2v^2 + 9656cv^2 + 12096c - 2529v^2 - 4032 + 64512c^5,$$

$$b_{3,num}^0 = 56448c^4 - 73215c^4v^2 + 112896c^3 - 23113c^3v^2 - 42336c^2 + 73215c^2v^2 - 40011cv^2 - 21168c + 10204v^2 + 7056 - 112896c^5,$$

$$b_{4,num}^0 = -225792c^4 + 325629c^4v^2 - 451584c^3 - 38624c^3v^2 - 96246c^2v^2 + 169344c^2 + 28968cv^2 + 84672c + 451584c^5 - 8047v^2 - 28224,$$

$$b_{5,num}^0 = -388196c^4v^2 + 282240c^4 + 564480c^3 - 27081c^3v^2 + 233349c^2v^2 - 211680c^2 - 111571cv^2 - 105840c + 28899v^2 - 564480c^5 + 35280.$$

Since for small values of  $v$ , the above formulas are subject to heavy cancellations, we give the Taylor expansions of  $b$  coefficients:

$$b_1 = \frac{399187}{241920} - \frac{52559}{912384} v^2 + \frac{100673687}{29059430400} v^4$$

$$- \frac{1084493}{27897053184} v^6 + \frac{96453547}{213412456857600} v^8,$$

$$b_2 = -\frac{17327}{8640} + \frac{52559}{114048} v^2 - \frac{100673687}{3632428800} v^4$$

$$+ \frac{1084493}{3487131648} v^6 - \frac{96453547}{26676557107200} v^8 + \dots,$$

$$b_3 = \frac{597859}{60480} - \frac{367913}{228096} v^2 + \frac{100673687}{1037836800} v^4$$

$$- \frac{1084493}{996323328} v^6 + \frac{96453547}{7621873459200} v^8 - \dots,$$

$$b_4 = -\frac{704183}{60480} + \frac{367913}{114048} v^2 - \frac{100673687}{518918400} v^4$$

$$+ \frac{1084493}{498161664} v^6 - \frac{96453547}{3810936729600} v^8 + \dots,$$

$$b_5 = \frac{465133}{24192} - \frac{1839565}{456192} v^2 + \frac{100673687}{415134720} v^4$$

$$- \frac{5422465}{1992646656} v^6 + \frac{96453547}{3048749383680} v^8 - \dots,$$

The PLTE of the method is given as

$$PLTE = \left( \frac{52559}{912384} w^2 y^{(10)} + \frac{52559}{912384} y^{(12)} \right) h^{12}. \quad (20)$$

### 3.3. Method PF-D1: Phase Fitted + First Derivative of Phase Lag is Zero

The second method of the family (PF-D1) is constructed by forcing the phase-lag function and its first derivative to be zero at frequency  $V = \omega \times h$ . The coefficients  $a$  are left the same,

while coefficients  $b$  become

$$b_0 = 0, \quad b_1 = \frac{1}{192} \frac{b_{1,num}^1}{D_1}, \quad b_2 = \frac{1}{48} \frac{b_{2,num}^1}{D_1},$$

$$b_3 = \frac{1}{48} \frac{b_{3,num}^1}{D_1}, \quad b_4 = \frac{1}{48} \frac{b_{4,num}^1}{D_1}, \quad b_5 = \frac{1}{96} \frac{b_{5,num}^1}{D_1},$$

where

$$D_1 = v^3((c)^5 - 3(c)^4 + 2(c)^3 + 2(c)^2 - 3c + w),$$

$$c = \cos(v), \quad s = \sin(v),$$

and

$$b_{1,num}^1 = -48s - 768c^3s + 144cs + 768sc^5 - 384c^4s + 288c^2s + 432cv - 941cv^3 + 281v^3 + 1344c^5v + 1344c^4v - 1776c^3v - 768c^6v - 576c^2v + 259c^2v^3 + 1481c^3v^3,$$

$$b_{2,num}^1 = 72s - 384sc^6 - 960sc^5 + 960c^4s + 1008c^3s - 504c^2s - 192cs - 1305c^3v^3 + 732c^2v + 427c^2v^3 + 225cv^3 - 612cv - 12v - 26v^3 + 192c^7v + 576c^6v - 2304c^5v - 1296c^4v - 1481v^3c^4 + 2724c^3v,$$

$$b_{3,num}^1 = -192s + 2304sc^6 + 1920sc^5 - 3840c^4s - 2208c^3s + 1584c^2s + 432cs - 7464c^3v + 3387c^3v^3 - 1512c^2v - 895c^2v^3 - 1088cv^3 + 1512cv + 72v - 1152c^7v + 384c^6v + 7104c^5v + 1481v^3c^5 + 4443v^3c^4 + 1056(c)^4v + 232v^3,$$

$$b_{4,num}^1 = -192s + 2304sc^6 + 1920sc^5 - 3840c^4s - 2208c^3s + 1584c^2s + 432cs - 7464c^3v + 3387c^3v^3 - 1512c^2v - 895c^2v^3 - 1088cv^3 + 1512cv + 72v - 1152c^7v + 384c^6v + 7104c^5v + 1481v^3c^5 + 4443v^3c^4 + 1056(c)^4v + 232v^3,$$

$$b_{5,num}^1 = -720s + 15360sc^6 + 3840sc^5 - 21120c^4s - 5760c^3s + 7200c^2s + 1200cs - 29040c^3v + 10059c^3v^3 - 3360c^2v - 4231c^2v^3 + 5040cv - 3411cv^3 + 480v - 7680c^7v + 11520c^6v + 31680c^5v + 12252v^3c^5 - 8640c^4v + 22484v^3c^4 + 647v^3.$$

Since for small values of  $v$ , the above formulas are subject to heavy cancellations, we give the Taylor expansions of the coefficients  $b$ :

$$b_1 = \frac{399187}{241920} - \frac{52559}{456192} v^2 + \frac{975124291}{174356582400} v^4$$

$$- \frac{2896813}{49816166400} v^6 + \frac{1818828019}{1067062284288000} v^8,$$

$$b_2 = -\frac{17327}{8640} + \frac{52559}{57024} v^2 - \frac{4461254807}{43589145600} v^4$$

$$+ \frac{2517959}{340540200} v^6 - \frac{98779707713}{266765571072000} v^8 + \dots,$$

$$b_3 = \frac{597859}{60480} - \frac{367913}{114048} v^2 + \frac{3127415341}{6227020800} v^4$$

$$- \frac{3766196569}{87178291200} v^6 + \frac{44891085091}{20520428544000} v^8 - \dots,$$

$$b_4 = -\frac{704183}{60480} + \frac{367913}{57024} v^2 - \frac{7330976207}{6227020800} v^4$$

$$+ \frac{584032469}{5448643200} v^6 - \frac{1452594367391}{266765571072000} v^8,$$

$$b_5 = \frac{465133}{24192} - \frac{1839565}{228096} v^2 + \frac{3844845691}{2490808320} v^4$$

$$- \frac{4974280813}{34871316480} v^6 + \frac{773868209533}{106706228428800} v^8.$$

The PLTE of the method is given as

$$PLTE = \left( \frac{52559}{456192} w^2 y^{(10)} + \frac{52559}{912384} w^4 y^{(8)} + \frac{52559}{912384} y^{(12)} \right) h^{12}. \quad (21)$$

3.4. Method PF-D2: Phase Fitted + First, Second Derivatives of Phase Lag are Zero

The third method of the family (PF-D2) is constructed by forcing the phase-lag function and its first and second derivatives to be zero at the frequency  $V = \omega \times h$ . The coefficients  $a$  are left the same, while for the coefficients  $b$  we have

$$b_0 = 0, \quad b_1 = \frac{1}{32} \frac{b_{1,num}^2}{D_2'}, \quad b_2 = \frac{1}{16} \frac{b_{2,num}^2}{D_2},$$

$$b_3 = \frac{1}{8} \frac{b_{3,num}^2}{D_2'}, \quad b_4 = \frac{1}{16} \frac{b_{4,num}^2}{D_2}, \quad b_5 = \frac{1}{16} \frac{b_{5,num}^2}{D_2'},$$

where

$$D_2 = v^4((c)^5 - 3(c)^4 + 2(c)^3 + 2(c)^2 - 3c + 1),$$

$$D_2' = v^4(s)^4((c)^2 - 2c + 1), \quad c = \cos(v), \quad s = \sin(v),$$

and

$$b_{1,num}^2 = -48s - 768c^3s + 144cs + 768sc^5 - 384c^4s + 288c^2s + 432cv - 941cv^3 + 281v^3 + 1344c^5v + 1344c^4v - 1776c^3v - 768c^6v - 576c^2v + 259c^2v^3 + 1481c^3v^3,$$

$$b_{2,num}^2 = -24 - 800vc^3s - 512svc^6 + 1088vc^4s - 25v^4 + 628c^3v^2 - 644c^2v^2 + 216vcs + 24vs + 168c^2 - 528vc^2s - 640c^5v^2 + 68v^2 + 768c^5 + 192c^6 - 180cv^2 + 385c^2v^4 + 72c - 456c^3 - 336c^4 + 512svc^5 + 435c^3v^4 + 768c^4v^2 - 384c^7 + 192v^2c^7 - 192v^2c^6 - 75cv^4,$$

$$b_{3,num}^2 = 24 + 1612vc^3s + 736svc^7 - 960svc^6 + 676vc^4s + 55v^4 + 548c^3v^2 + 804c^2v^2 - 240vcs - 32vs + 192sc^8v - 1050c^4v^4 + 672c^8 - 300c^2 + 136vc^2s - 1028c^5v^2 - 60v^2 + 132c^5 - 1560c^6 - 435v^4c^5 + 96cv^2 - 256c^8v^2 - 265c^2v^4 - 24c + 84c^3 + 1164c^4 - 2120svc^5 - 865c^3v^4 - 1264c^4v^2 - 64c^9v^2 - 384c^7 + 192c^9 + 448v^2c^7 + 776v^2c^6 + 40cv^4,$$

$$b_{4,num}^2 = -72 - 3904vc^3s - 1536svc^7 + 256svc^6 + 512vc^4s - 75v^4 - 244c^3v^2 - 1980c^2v^2 + 648vcs + 104vs + 1740c^4v^4 - 1536c^8 + 792c^2 - 944vc^2s + 832c^5v^2 + 156v^2 + 960c^5 + 3648c^6 + 580v^4c^5 - 396cv^2 + 512c^8v^2 + 855c^2v^4 + 120c - 696c^3 - 2832c^4 + 4864svc^5 + 2165c^3v^4 + 3168c^4v^2 - 384c^7 - 192v^2c^7 - 1856v^2c^6 - 225cv^4,$$

$$b_{5,num}^2 = 84 + 5416vc^3s + 640svc^7 - 6720svc^6 + 4080vc^4s + 165v^4 + 2540c^3v^2 + 2764c^2v^2 - 900vcs - 120vs + 2304sc^8v - 4720c^4v^4 + 1920c^8 - 1020c^2 + 420vc^2s - 920v^4c^6 - 4528c^5v^2 - 172v^2 + 2736c^5 - 4800c^6 - 3380v^4c^5 + 260cv^2 - 825c^2v^4 - 60c - 180c^3 + 3816c^4 - 5120svc^5 - 3115c^3v^4 - 2912c^4v^2 - 768c^9v^2 - 4800c^7 + 2304c^9 + 2496v^2c^7 + 320v^2c^6 + 195cv^4.$$

Since for small values of  $v$ , the above formulas are subject to heavy cancellations, we give the Taylor expansions of the coefficients  $b$ :

$$b_1 = \frac{399187}{241920} - \frac{52559}{304128} v^2 + \frac{371082169}{58118860800} v^4$$

$$- \frac{83360891}{523069747200} v^6 - \frac{1467578899}{355687428096000} v^8,$$

$$b_2 = -\frac{17327}{8640} + \frac{52559}{38016} v^2 - \frac{3253170563}{14529715200} v^4$$

$$+ \frac{5070942803}{261534873600} v^6 - \frac{86978398867}{88921857024000} v^8,$$

$$b_3 = \frac{597859}{60480} - \frac{367913}{76032} v^2 + \frac{2523373219}{2075673600} v^4$$

$$- \frac{22329042629}{130767436800} v^6 + \frac{1453392734357}{88921857024000} v^8,$$

$$b_4 = -\frac{704183}{60480} + \frac{367913}{38016} v^2 - \frac{6122891963}{2075673600} v^4$$

$$+ \frac{133660742933}{261534873600} v^6 - \frac{5024895032029}{88921857024000} v^8,$$

$$b_5 = \frac{465133}{24192} - \frac{1839565}{152064} v^2 + \frac{3240803569}{830269440} v^4$$

$$- \frac{37612768013}{52306974720} v^6 + \frac{2927078073011}{35568742809600} v^8.$$

The PLTE of the method is given as

$$PLTE = \left( \frac{52559}{912384} w^6 y^{(6)} + \frac{52559}{304128} w^2 y^{(10)} + \frac{52559}{912384} y^{(12)} \right. \\ \left. + \frac{52559}{304128} w^4 y^{(8)} \right) h^{12}. \tag{22}$$

3.5. Method PF-D3: Phase Fitted + First, Second, and Third Derivatives of Phase Lag are Zero

The fourth method of the family (PF-D3) is constructed by forcing the phase-lag function and its first, second, and third derivatives to be zero at the frequency  $V = \omega \times h$ . The coefficients  $a$  are left the same, while for the coefficients  $b$  we have

$$b_0 = 0, \quad b_1 = \frac{1}{48} \frac{b_{1,num}^3}{D_3}, \quad b_2 = \frac{1}{12} \frac{b_{2,num}^3}{D_3},$$

$$b_3 = \frac{1}{12} \frac{b_{3,num}^3}{D_3}, \quad b_4 = \frac{1}{12} \frac{b_{4,num}^3}{D_3}, \quad b_5 = \frac{1}{24} \frac{b_{5,num}^3}{D_3},$$

where

$$D_3 = v^5(c^7 - c^6 - 3c^5 + 3c^4 + 3c^3 - 3c^2 - c + 1),$$

$$c = \cos(v), \quad s = \sin(v),$$

and

$$b_{1,num}^3 = -2096v^2c^5s + 1508v^2c^4s + 1648v^2c^3s - 964v^2s^2c^2 + 36v - 366v^2cs + 135v^5c - 48v^3 - 384c^8v^3 - 336c^4s - 656v^2c^6s - 336cv^3 + 1440c^5v - 738c^2v - 384sc^7 + 168c^2s + 234cv + 528c^2v^3 - 1098c^3v - 2448c^6v + 768sc^5 + 192sc^6 - 456c^3s + 1008c^3v^3 + 72cs + 864vc^8 + 2286c^4v + 45v^3c^3 - 24s + 1200c^6v^3 - 1008c^5v^3 + 45v^5 + 135v^5c^2 - 576c^7v - 1296c^4v^3 + 832sv^2c^7 + 94v^2s + 336c^7v^3,$$

$$b_{2,num}^3 = y - 692v^2c^5s - 2407v^2c^4s + 235v^2c^3s + 776v^2sc^2 - 27v + 123v^2cs - 90v^5c + 48v^3 - 96c^8v^3 + 852c^4s + 2560v^2c^6s + 144cv^3 - 3690c^5v + 90c^2v - 96sc^7 - 192c^2s - 144cv - 48c^2v^3 + 1458c^3v - 738c^6v + 120sc^5 - 1248sc^6 - 24c^3s - 912sv^2c^8 - 720c^3v^3 + 360vc^8 + 315c^4v + 576sc^8 - 270v^5c^3 - 1152c^9v + 12s + 240c^6v^3 + 1296c^5v^3 - 270v^5c^2 - 90c^4v^5 + 3528c^7v - 144c^4v^3 + 352sv^2c^7 + 288c^9v^3 - 35v^2s - 1008c^7v^3,$$

$$b_{3,num}^3 = 3664v^2c^5s + 3218v^2c^4s - 812v^2c^3s - 1498v^2sc^2 + 54v - 258v^2cs + 135v^5c - 84v^3 + 1344c^8v^3 - 1200c^4s - 2420v^2c^6s + 1344sv^2c^9 - 300cv^3 + 4842c^5v - 180c^2v + 2496sc^7 - 384c^{10}v^3 + 312c^2s + 324cv + 60c^2v^3 - 2502c^3v + 6498c^6v - 1608sc^5 + 270c^5v^5 + 1488sc^6 + 264c^3s + 624sv^2c^8 + 1092c^3v^3 - 6336vc^8 - 2052c^4v - 576sc^8 + 855v^5c^3 + 864c^9v - 24s - 1644c^6v^3 - 1476c^5v^3 + 45v^5 + 405v^5c^2 - 1152sc^9 + 810c^4v^5 - 3528c^7v + 708c^4v^3 - 3920sv^2c^7 - 192c^9v^3 + 2016vc^{10} + 58v^2s + 876c^7v^3,$$

$$b_{4,num}^3 = -5752v^2c^5s - 3789v^2c^4s + 1333v^2c^3s + 2196v^2sc^2 - 81v + 417v^2cs - 270v^5c + 108v^3 - 2016c^8v^3 + 1404c^4s + 1036v^2c^6s - 2048sv^2c^9 + 468cv^3 - 5310c^5v + 234c^2v - 4224sc^7$$

$$\begin{aligned}
 &+576 c^{10} v^3 - 432 c^2 s - 504 c v - 36 c^2 v^3 + 3582 c^3 v - 10818 c^6 v \\
 &+ 2760 s c^5 - 1080 c^5 v^5 - 816 s c^6 - 456 c^3 s + 1360 s v^2 c^8 - 1500 c^3 v^3 \\
 &- 1152 c^{11} v + 10224 v c^8 + 3609 c^4 v - 960 s c^8 - 1170 v^5 c^3 + 2592 c^9 v \\
 &+ 36 s + 768 s c^{10} + 2484 c^6 v^3 + 1500 c^5 v^3 + 192 c^{11} v^3 - 810 v^5 c^2 \\
 &- 704 s v^2 c^{10} + 1920 s c^9 - 1350 c^4 v^5 + 792 c^7 v - 1116 c^4 v^3 + 6032 \\
 &s v^2 c^7 - 360 v^5 c^6 - 480 c^9 v^3 - 3168 v c^{10} - 81 v^2 s - 180 c^7 v^3, \\
 &b_{3,\text{num}}^3 = 13216 v^2 c^5 s + 10404 v^2 c^4 s - 4672 v^2 c^3 s - 4932 v^2 \\
 &s c^2 + 180 v - 762 v^2 c s + 405 v^5 c - 240 v^3 + 3456 c^8 v^3 - 3888 c^4 s \\
 &- 4048 v^2 c^6 s + 2816 s v^2 c^9 - 912 c v^3 + 15192 c^5 v + 162 c^2 v + 7680 \\
 &s c^7 - 768 c^{10} v^3 + 1080 c^2 s + 1062 c v - 432 c^2 v^3 - 9054 c^3 v \\
 &+ 22680 c^6 v - 5856 s c^5 + 2160 c^5 v^5 + 2112 s c^6 + 1320 c^3 s - 4288 \\
 &s v^2 c^8 + 3504 c^3 v^3 + 4608 c^{11} v - 72 c s - 17856 v c^8 - 9774 c^4 v \\
 &+ 3840 s c^8 + 3375 v^5 c^3 - 9216 c^9 v - 72 s - 3072 s c^{10} - 5520 c^6 v^3 \\
 &- 4272 c^5 v^3 + 135 v^5 - 768 c^{11} v^3 + 1485 v^5 c^2 + 2816 s v^2 c^{10} - 3072 \\
 &>\text{breaksc}^9 + 3600 c^4 v^5 - 2592 c^7 v + 3504 c^4 v^3 - 10688 s v^2 c^7 \\
 &+ 360 v^5 c^7 + 1080 v^5 c^6 + 1536 c^9 v^3 + 4608 v c^{10} + 138 v^2 s + 912 c^7 v^3.
 \end{aligned}$$

Since for small values of  $v$ , the above formulas are subject to heavy cancellations, we give the Taylor expansions of the coefficients  $b$ :

$$\begin{aligned}
 b_1 &= \frac{399187}{241920} - \frac{52559}{228096} v^2 + \frac{11315653}{1937295360} v^4 \\
 &\quad - \frac{5807033}{13076743680} v^6 - \frac{614853845}{17072996548608} v^8, \\
 b_2 &= -\frac{17327}{8640} + \frac{52559}{28512} v^2 - \frac{5758537}{14676480} v^4 \\
 &\quad + \frac{225159101}{6538371840} v^6 - \frac{8200289261}{4268249137152} v^8, \\
 b_3 &= \frac{597859}{60480} - \frac{367913}{57024} v^2 + \frac{154801723}{69189120} v^4 \\
 &\quad - \frac{2799488011}{6538371840} v^6 + \frac{1063054198007}{21341245685760} v^8, \\
 b_4 &= -\frac{704183}{60480} + \frac{367913}{28512} v^2 - \frac{381346481}{69189120} v^4 \\
 &\quad + \frac{9217976399}{6538371840} v^6 - \frac{5100295346143}{21341245685760} v^8, \\
 b_5 &= \frac{465133}{24192} - \frac{1839565}{114048} v^2 + \frac{67543471}{9225216} v^4 \\
 &\quad - \frac{241481599}{118879488} v^6 + \frac{16316044646989}{42682491371520} v^8.
 \end{aligned}$$

The PLTE of the method is given as

$$\begin{aligned}
 \text{PLTE} &= \left( \frac{52559}{228096} w^2 y^{(10)} + \frac{52559}{912384} w^8 y^{(4)} + \frac{52559}{152064} w^4 y^{(8)} \right. \\
 &\quad \left. + \frac{52559}{228096} w^6 y^{(6)} + \frac{52559}{912384} y^{(12)} \right) h^{12}.
 \end{aligned}$$

### 3.6. Method PF-D4: Phase Fitted + First, Second, Third, and Fourth Derivatives of Phase Lag are Zero

The fifth method of the family (PF-D4) is constructed by forcing the phase-lag function and its first, second, third, and fourth derivatives to be zero at the frequency  $V = \omega \times h$ . The coefficients  $a$  are left the same, while for the coefficients  $b$  we have

$$\begin{aligned}
 b_0 &= 0, & b_1 &= \frac{1}{96} \frac{b_{1,\text{num}}^4}{D_4}, & b_2 &= \frac{1}{12} \frac{b_{2,\text{num}}^4}{D_4}, \\
 b_3 &= \frac{1}{24} \frac{b_{3,\text{num}}^4}{D_4}, & b_4 &= \frac{1}{12} \frac{b_{4,\text{num}}^4}{D_4}, & b_5 &= \frac{1}{48} \frac{b_{5,\text{num}}^4}{D_4},
 \end{aligned}$$

where

$$D_4 = v^6(c+1)(s)^5, \quad c = \cos(v), \quad s = \sin(v),$$

and

$$\begin{aligned}
 b_{1,\text{num}}^4 &= 3069 s c^2 v^2 - 6408 s c^4 v^2 - 4218 s c^3 v^2 - 864 s c v^4 + \\
 &1728 s c^3 v^4 + 2016 s c^4 v^4 - 864 s c^5 v^4 + 1380 s c v^2 - 1152 s c^2 v^4 + \\
 &2568 s c^5 v^2 + 3408 s c^6 v^2 - 960 s c^6 v^4 - 144 v + 1440 c^4 s + \\
 &1029 c v^3 + 60 s + 372 v^3 + 4836 c^4 v^3 - 5952 c^5 v + 1728 c^2 v - \\
 &540 c^2 s - 648 c v - 3234 c^2 v^3 + 3912 c^3 v + 1728 c^6 v - 480 s c^5 + \\
 &2688 c^7 v - 2064 c^6 v^3 + 600 c^3 s - 204 s v^2 - 4786 c^3 v^3 + 96 s v^4 - \\
 &120 c s - 3312 c^4 v + 6176 c^5 v^3 - 960 s c^6 - 2464 c^7 v^3, \\
 b_{2,\text{num}}^4 &= -1047 s c^2 v^2 + 3552 s c^4 v^2 - 2295 s c^3 v^2 + 48 s c v^4 + \\
 &384 s c^3 v^4 - 912 s c^4 v^4 - 912 s c^5 v^4 + 78 s c v^2 + 384 s c^2 v^4 + \\
 &5184 s c^5 v^2 - 2208 s c^6 v^2 - 2832 s v^2 c^7 + 480 s c^6 v^4 + 12 v - \\
 &600 c^4 s - 131 c v^3 - 35 v^3 + 2929 c^4 v^3 + 3096 c^5 v + 528 c^2 v + \\
 &120 c^2 s + 120 c v - 433 c^2 v^3 - 1584 c^3 v + 5472 c^6 v - 1440 s c^5 - \\
 &1632 c^7 v - 4128 c^6 v^3 + 540 c^3 s - 27 s v^2 + 2181 c^3 v^3 + 48 s v^4 - \\
 &60 c s - 3516 c^4 v - 3480 c^5 v^3 + 480 s c^6 + 1520 c^7 v^3 + 1712 c^8 v^3 - \\
 &2496 c^8 v + 960 s c^7 + 480 s c^7 v^4, \\
 b_{3,\text{num}}^4 &= 2997 s c^2 v^2 + 3906 s c^4 v^2 + 498 s c^3 v^2 - 936 s c v^4 - \\
 &48 s c^3 v^4 + 24 s c^4 v^4 + 2904 s c^5 v^4 + 1632 s c v^2 - 1008 s c^2 v^4 - \\
 &15396 s c^5 v^2 - 21792 s c^6 v^2 - 7488 c^9 v^3 + 11376 s v^2 c^7 + \\
 &2880 s c^6 v^4 - 144 v - 1800 c^4 s + 1101 c v^3 + 60 s + 300 v^3 - \\
 &4740 c^4 v^3 + 12768 c^5 v + 1152 c^2 v - 180 c^2 s - 792 c v - 2910 c^2 v^3 + \\
 &1464 c^3 v - 15552 c^6 v + 3120 s c^5 - 27264 c^7 v + 13632 c^6 v^3 - \\
 &120 c^3 s - 168 s v^2 - 3244 c^3 v^3 + 24 s v^4 - 120 c s + 5328 c^8 v - \\
 &5980 c^5 v^3 + 7680 s c^6 + 15296 c^7 v^3 - 6912 c^8 v^3 + 9216 c^8 v - \\
 &2880 s c^7 + 14112 s c^8 v^2 + 13824 c^9 v - 1920 s c^7 v^4 - 1920 s c^8 v^4 - \\
 &5760 s c^8, \\
 b_{4,\text{num}}^4 &= -3321 s c^2 v^2 + 8268 s c^4 v^2 - 6861 s c^3 v^2 + 108 s c v^4 + \\
 &1224 s c^3 v^4 - 1812 s c^4 v^4 - 1812 s c^5 v^4 + 162 s c v^2 + 1224 s c^2 v^4 + \\
 &10188 s c^5 v^2 + 3552 s c^6 v^2 + 960 s c^9 v^4 + 3648 c^9 v^3 + 5328 s v^2 c^7 - \\
 &7872 s v^2 c^9 - 480 s c^6 v^4 + 36 v - 1320 c^4 s - 297 c v^3 - 81 v^3 + \\
 &7815 c^4 v^3 + 4104 c^5 v + 1728 c^2 v + 360 c^2 s + 360 c v - 1287 c^2 v^3 - \\
 &4464 c^3 v + 5520 c^6 v - 2160 s c^5 + 5760 c^7 v - 5828 c^6 v^3 + \\
 &1380 c^3 s - 81 s v^2 + 6315 c^3 v^3 + 108 s v^4 - 180 c s - 9396 c^4 v - \\
 &5244 c^5 v^3 - 960 s c^6 - 3792 c^7 v^3 + 3840 s c^9 + 3904 c^{10} v^3 - \\
 &4208 c^8 v^3 + 10560 c^8 v - 2880 s c^7 - 6528 s c^8 v^2 - 5760 c^9 v - \\
 &480 s c^7 v^4 + 960 s c^8 v^4 + 1920 s c^8 - 8448 c^{10} v, \\
 b_{5,\text{num}}^4 &= 9207 s c^2 v^2 + 21552 s c^4 v^2 + 6498 s c^3 v^2 - 2784 s c v^4 - \\
 &2112 s c^3 v^4 - 1824 s c^4 v^4 + 11040 s c^5 v^4 + 5148 s c v^2 - 3072 s c^2 v^4 - \\
 &60312 s c^5 v^2 - 81288 s c^6 v^2 - 1536 s c^9 v^4 - 14592 c^9 v^3 + \\
 &27888 s v^2 c^7 + 11328 s v^2 c^9 + 10944 s c^6 v^4 - 432 v - 8160 c^4 s + \\
 &3183 c v^3 + 180 s + 828 v^3 - 23348 c^4 v^3 + 55680 c^5 v + 2880 c^2 v - \\
 &180 c^2 s - 2520 c v - 8598 c^2 v^3 + 1560 c^3 v - 54720 c^6 v + \\
 &12000 s c^5 - 93120 c^7 v + 47712 c^6 v^3 - 1080 c^3 s - 468 s v^2 - \\
 &7742 c^3 v^3 + 96 s v^4 - 360 c s + 24240 c^4 v - 29032 c^5 v^3 + \\
 &27360 s c^6 + 53008 c^7 v^3 - 3840 s c^9 - 6016 c^{10} v^3 - 13728 c^8 v^3 + \\
 &17280 c^8 v - 6720 s c^7 + 32832 s c^8 v^2 + 23040 c^9 v - 6400 c^{11} v^3 + \\
 &13440 s c^{10} v^2 - 4608 s c^7 v^4 - 4608 s c^8 v^4 - 11520 s c^8 - \\
 &7680 s c^{10} + 10752 c^{10} v - 1536 s c^{10} v^4 + 15360 c^{11} v.
 \end{aligned}$$

Since for small values of  $v$ , the above formulas are subject to heavy cancellations, we give the Taylor expansions of the coefficients  $b$ :

$$\begin{aligned}
 b_1 &= \frac{399187}{241920} - \frac{262795}{912384} v^2 + \frac{17265277}{4358914560} v^4 \\
 &\quad - \frac{38566679}{38041436160} v^6 - \frac{52935007231}{426824913715200} v^8, \\
 b_2 &= -\frac{17327}{8640} + \frac{262795}{114048} v^2 - \frac{2649128441}{4358914560} v^4
 \end{aligned}$$

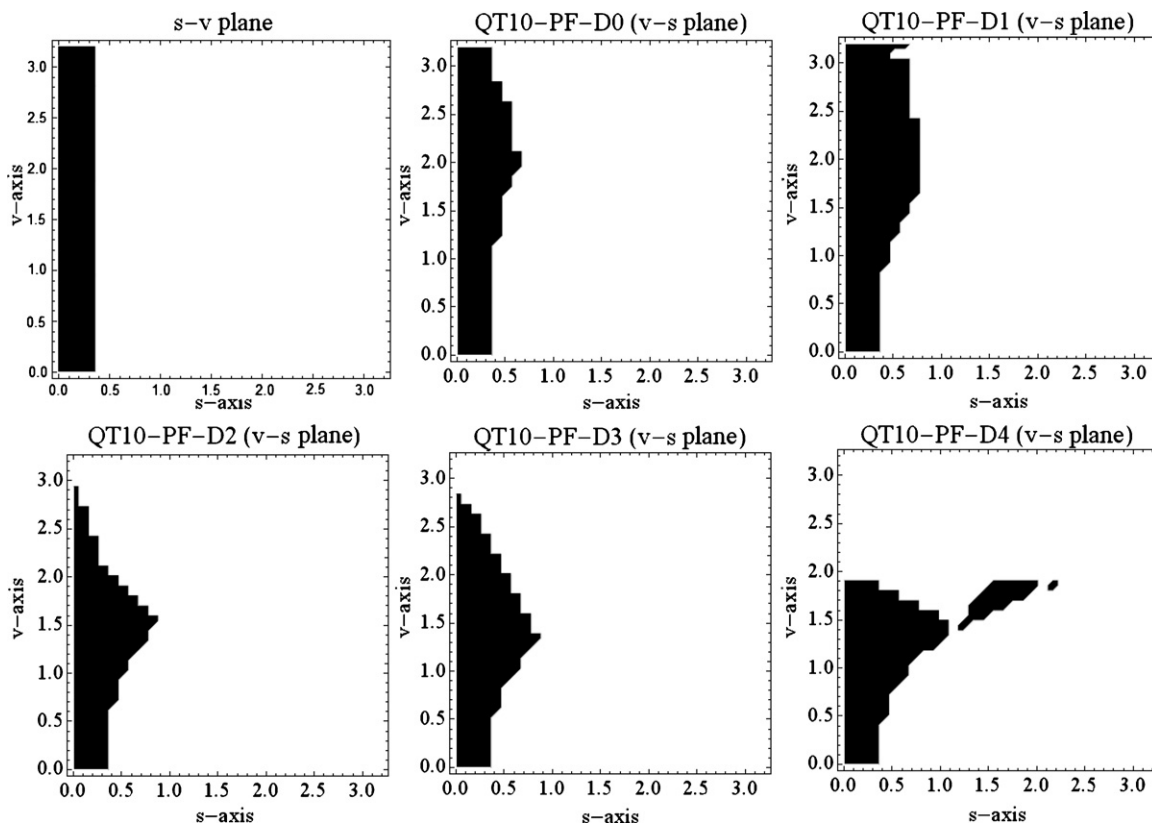


Figure 1. Stability regions (shaded areas) for the six methods.

$$\begin{aligned}
 & + \frac{2650726483}{52306974720} v^6 - \frac{374485131133}{106706228428800} v^8, \\
 b_3 = & \frac{597859}{60480} - \frac{1839565}{228096} v^2 + \frac{1110676079}{311351040} v^4 \\
 & - \frac{6919361527}{8047226880} v^6 + \frac{1637603830619}{15243746918400} v^8, \\
 b_4 = & -\frac{704183}{60480} + \frac{1839565}{114048} v^2 - \frac{5518849841}{622702080} v^4 \\
 & + \frac{14263211663}{4755179520} v^6 - \frac{73005517242211}{106706228428800} v^8, \\
 b_5 = & \frac{465133}{24192} - \frac{9197825}{456192} v^2 + \frac{734695627}{62270208} v^4 \\
 & - \frac{183227481067}{41845579776} v^6 + \frac{9908801489731}{8536498274304} v^8,
 \end{aligned}$$

The PLTE of the method is given as

$$\begin{aligned}
 \text{PLTE} = & \left( \frac{52559}{912384} w^{10} y^{(2)} + \frac{262795}{456192} w^6 y^{(6)} \right. \\
 & + \frac{262795}{912384} w^8 y^{(4)} + \frac{262795}{456192} w^4 y^{(8)} \\
 & \left. + \frac{52559}{912384} y^{(12)} + \frac{262795}{912384} w^2 y^{(10)} \right) h^{12}.
 \end{aligned}$$

#### 4. STABILITY ANALYSIS

The stability of the new methods is studied by considering the test equation

$$\frac{d^2 y(t)}{dt^2} = -\sigma^2 y(t) \tag{23}$$

and the linear multistep method (3) for the numerical solution. In the above equation,  $\sigma \neq \omega$  ( $\omega$  is the frequency at which the phase-lag function and its derivatives vanish). Setting  $s = \sigma h$  and  $v = \omega h$ , we get for the characteristic equation of the applied method

$$\sum_{j=1}^{J/2} A_j(s^2, v)(z^j + z^{-j}) + A_0(s^2, v) = 0, \tag{24}$$

where

$$A_j(s^2, v) = a_{\frac{j}{2}-j}(v) + s^2 \cdot b_{\frac{j}{2}-j}(v). \tag{25}$$

The motivation of the above analysis is straightforward: although the coefficients of method (3) are designed in a way that the phase lag and its first derivatives vanish in the frequency  $\omega$ , the frequency  $\omega$  itself is unknown and only an estimation can be made. Thus, if the correct frequency of the problem is  $\sigma$  we have to check if the method is stable, that is if the roots of the characteristic equation lie on the unit disk. For this reason, we draw at the  $v$ - $s$  plane the areas in which the method is stable. Figure 1 shows the stability region for the six methods (the classical one, the phase fitted one, and those with first, second, third, and fourth phase-lag derivative elimination). Note here that the  $s$ -axis corresponds to the real frequency, while the  $v$ -axis corresponds to the estimated frequency used to construct the parameters of the method.

#### 5. NUMERICAL RESULTS

Numerical experiments have been carried out for two orbital problems. Since the classical method is well studied, we only present the new methods in comparison to the classical one.

5.1. The 2-Body Problem

In this problem, we test the motion of two bodies in a reference system that is fixed in one of them. Moreover, the motion is planar, thus, we only have to calculate the  $x$  and  $y$  coordinates of the second body. The differential equations are

$$\ddot{x} = -\frac{x}{(x^2 + y^2)^3},$$

$$\ddot{y} = -\frac{y}{(x^2 + y^2)^3},$$

and the initial conditions are

$$x(0) = 1 - \epsilon, \quad \dot{x}(0) = 0,$$

$$y(0) = 0, \quad \dot{y}(0) = \sqrt{\frac{1 + \epsilon}{1 - \epsilon}},$$

where  $\epsilon$  is the eccentricity. The theoretical solution is given below:

$$x(t) = \cos(u) - \epsilon, \tag{26}$$

$$y(t) = \sqrt{1 - \epsilon^2} \sin(t),$$

where  $u$  can be found by solving the equation  $u - \epsilon \sin(u) - t = 0$ .

We used an estimation for the frequency  $\omega = 1$ .

Figures 5–8 present the accuracy of the methods expressed by  $-\log_{10}(\text{ERR})$  versus the  $\log_{10}$  (total steps) for eccentricities  $\epsilon = 0.001, 0.1, 0.5,$  and  $0.9,$  respectively. ERR is the maximum absolute error of the position coordinates  $x$  and  $y$  over the integration interval, that is we compare the approximate solutions  $x$  and  $y$  to the theoretical solution of  $x$  and  $y$  at each point of integration and take the maximum of the absolute differences.

5.2. The Five-Outer Planet System

The next problem concerns the motion of the five outer planets relative to the Sun. The problem falls in the category of the  $N$ -Body problem which is the problem that regards the movement of  $N$  bodies under Newton’s law of gravity. It is expressed by a system of vector differential equations

$$\ddot{\mathbf{y}}_i = G \sum_{j=1, j \neq i}^N \frac{m_j(\mathbf{y}_j - \mathbf{y}_i)}{|\mathbf{y}_j - \mathbf{y}_i|^3}, \tag{27}$$

where  $G$  is the gravitational constant,  $m_j$  is the mass of body  $j$ , and  $\mathbf{y}_i$  is the vector of the position of body  $i$ . It is easy to see that each vector differential Equation of (29) can be analyzed into three simplified differential equations, that express the three directions  $x, y, z$ .

The above system of ODEs cannot be solved analytically. Instead we produce a highly accurate numerical solution by using a 10-stage implicit Runge–Kutta method of Gauss with 20th algebraic order, that is also symplectic and A-stable. The method can be easily reproduced using simplifying assumptions for the order conditions (see Butcher 2003). The reference solution is obtained by using the previous method to integrate the  $N$ -body problem for a specific time-span and for different step-lengths.

In Hairer et al. (2002), the data for the five-outer planet problem are given (these data are summarized in Table 1). Masses are relative to the Sun, so that the Sun has mass 1. In the computations, the Sun with the four inner planets are considered one body, so the mass is larger than one. Distances are in astronomical units, time is in Earth days, and

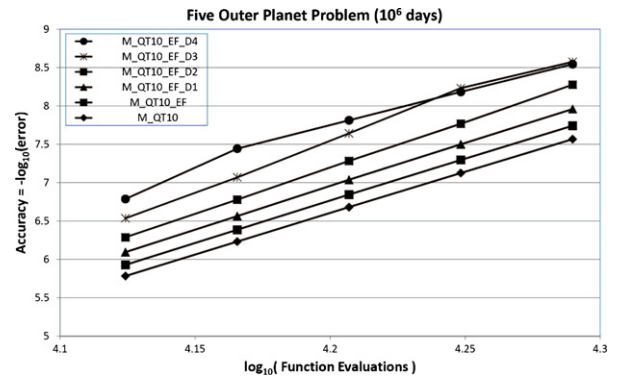


Figure 2. Five-outer planet problem for  $10^6$  days.

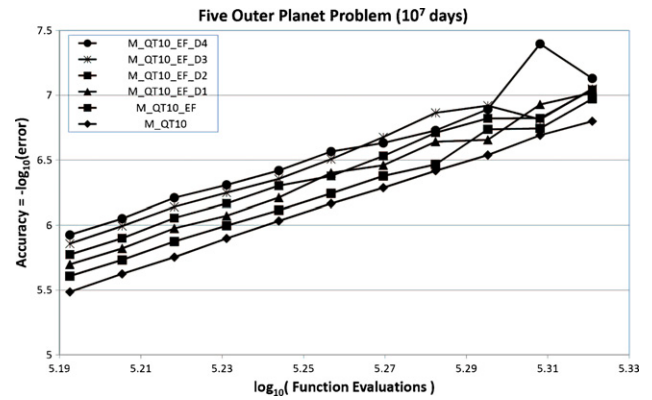


Figure 3. Five-outer planet problem for  $10^7$  days.

Table 1  
Initial Data for the Five-Outer Planet Problem

Planet	Mass	Initial Position	Initial Velocity
Sun	1.00000597682	0 0 0	0 0 0
Jupiter	0.000954786104043	-3.5023653 -3.8169847 -1.5507963	0.00565429 -0.00412490 -0.00190589
Saturn	0.000285583733151	9.0755314 -3.0458353 -1.6483708	0.00168318 0.00483525 0.00192462
Uranus	0.0000437273164546	8.3101420 -16.2901086 -7.2521278	0.00354178 0.00137102 0.00055029
Neptune	0.0000517759138449	11.4707666 -25.7294829 -10.8169456	0.00288930 0.00114527 0.00039677
Pluto	$1/(1.3 \times 10^8)$	-15.5387357 -25.2225594 -3.1902382	0.00276725 -0.00170702 -0.00136504

the gravitational constant is  $G = 2.95912208286 \times 10^{-4}$ . The system of Equations (27) has been solved for  $10^6$  and  $10^7$  days, and for these time-spans, the previously mentioned method of Gauss produces a 10.5 and an 8.6 decimal digits accurate solution, respectively. Figures 2 and 3 present the accuracy of the methods expressed by  $-\log_{10}(\text{ERR})$  versus the  $\log_{10}$  (total steps). ERR is the error at the end point of integration, that is we compare the three position coordinates of all six bodies to the corresponding ones of the reference solution and we take the maximum of the absolute differences.

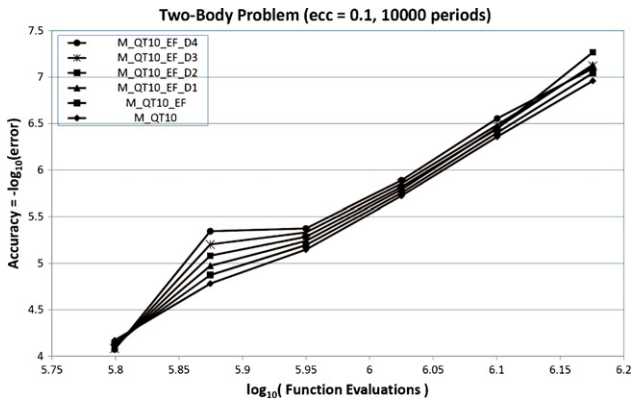


Figure 4. Two-body problem for eccentricity  $\epsilon = 0.1$  and 10,000 periods.

As for the frequency, we use the constant value 0.00145044732989. This is the frequency in radians of the fastest (in terms of angular speed) body, which is Jupiter. During the time span of  $10^7$  days (27,378 years), Jupiter rotates approximately 2308 times, while Pluto rotates approximately 110 times.

5.3. Results

In Figure 1, we see the stability regions ( $v-s$  plane), shown as the shadowed areas, for the classical Quinlan–Tremaine method, the PF-D0, PF-D1, PF-D2, PF-D3, and PF-D4 methods (from left to right and from top to bottom). We can confirm the increase of the stability interval ( $v = s$  diagonal) as the level of tuning increases, that is when more derivatives of the phase lag are nullified.

In Figure 2, we see the accuracy of the methods expressed by  $-\log_{10}$  (error at the end point which is after  $10^6$  days) versus the  $\log_{10}$  (total steps) for the five-outer planet system (see the previous subsection for explanation on the error computation). We can clearly observe the higher efficiency of the methods with high level of tuning over the methods with lower level. The new method with the phase lag and its four derivatives nullified has at least one digit higher accuracy than the classical Quinlan–Tremaine method. It is also correct to say that the new method needs a step-length approximately 20% smaller than that of the classical one, in order to produce a solution of the same accuracy. In this problem, as well as in all problems, we present the methods as soon as they converge (as we move to the right on the function evaluations axis, where  $h$  is decreasing). In this problem they all start at approximately  $h < 80$ . When the methods converge, they present high accuracy, due to their high algebraic order and the ability to behave well over a relative long integration.

In Figure 3, which presents a longer integration of the five-outer planet problem (Jupiter rotates 2308 times), we notice again the higher efficiency of the new methods in relation to the classical one. The nonlinearity, as we move to the right on the efficiency graphs, is observed because the accuracy approaches the accuracy of the reference solution (8.6 decimal points), thus the round-off error becomes important as compared to the truncation error of the methods.

In Figure 4, we perform a long time integration of the two-body problem for eccentricity  $\epsilon = 0.1$ . The time span is  $[0, 63000]$ , which corresponds to more than 10,000 periods. We can see a small but steady (along the function evaluations axis) difference in favor of the higher level of tuning methods. Every higher level of tuning method performs better than any

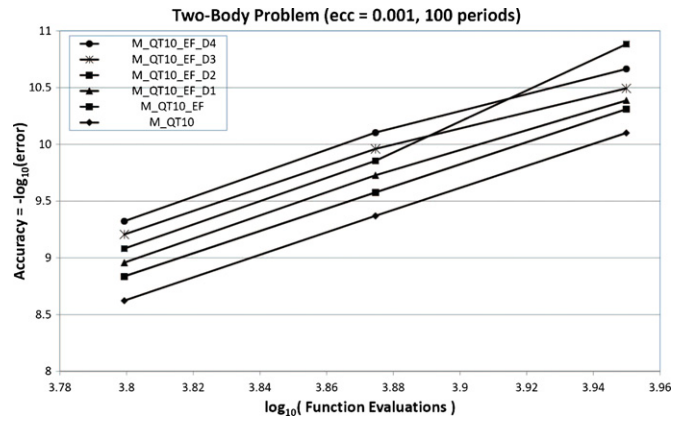


Figure 5. Two-body problem for eccentricity  $\epsilon = 0.001$  and 100 periods.

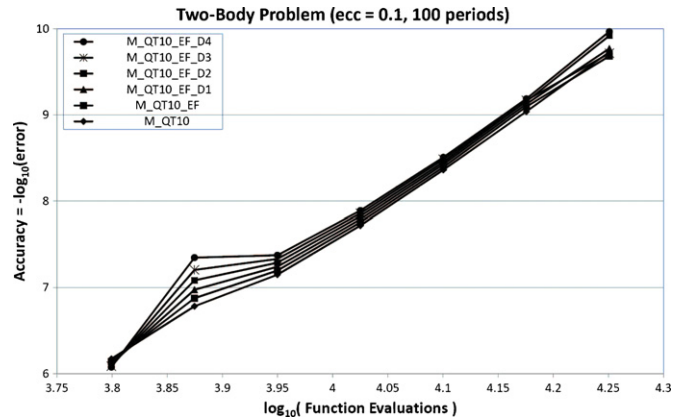


Figure 6. Two-body problem for eccentricity  $\epsilon = 0.1$  and 100 periods.

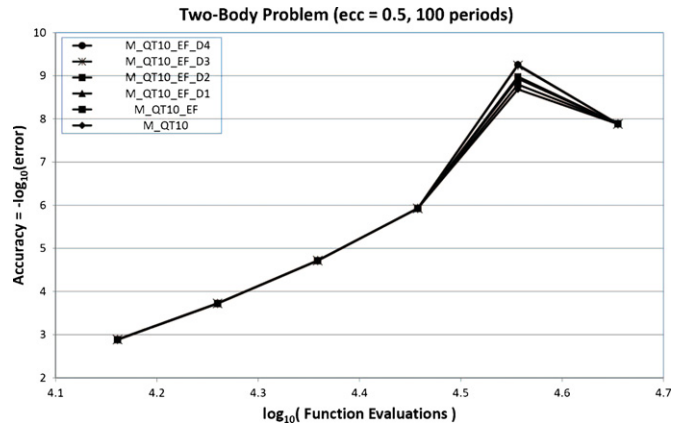


Figure 7. Two-body problem for eccentricity  $\epsilon = 0.5$  and 100 periods.

other method with lower level of tuning. All optimized methods are better than the classical Quinlan–Tremaine method and the new method with phase lag and four derivatives nullified has up to one digit higher accuracy than the classical one. We have also measured the energy loss during this integration. Since no extra energy-conservation property has been added to the classical methods, all six methods for all step-lengths tested had almost the same energy error: approximately 0.0079 (2.1000 digits of accuracy).

In Figures 5–8, we see the accuracy of the methods expressed by  $-\log_{10}$  (maximum error over the integration interval) versus the  $\log_{10}$  (total steps) in the two-body problem for eccentricities  $\epsilon = 0.001, 0.1, 0.5,$  and  $0.9$ . All four diagrams present the methods with the least necessary function evaluations (that



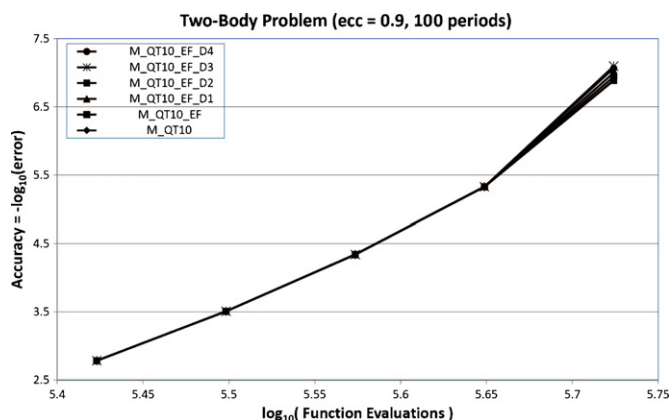


Figure 8. Two-body problem for eccentricity  $\epsilon = 0.9$  and 100 periods.

is with the highest step-length), which means that they did not converge with higher step-length (more left on the horizontal axis). The “starting” accuracy depends on the method’s properties (algebraic order and phase-lag error), the step-length used, and the length of the interval of integration. We can see that the methods perform a lot better on low values of eccentricity, while they perform almost similarly to the classical Quinlan–Tremaine method for higher eccentricities. This is explained by the fact that we added a better “phase-lag error behavior,” which helps a lot in orbital problems with low eccentricities and all oscillatory nonstiff problems, but little or less in other types of problems. We have also run the problem for  $\epsilon = 0.2, 0.3$ , etc., and the results showed that no significant improvement is observed above  $\epsilon = 0.2$ . In any case, the optimized methods performed better than the classical Quinlan–Tremaine method.

We note here that the function evaluations needed for the integration of each method are proportional to the CPU time needed, since the methods need the same function evaluations per step (one) and the overhead time of computing the variable coefficients is minimal, since we use a constant frequency.

## 6. CONCLUSIONS

We have presented a new family of 10-step symmetric multistep numerical methods with improved characteristics concerning orbital problems. The methods were constructed by adopting a new methodology which, apart from the phase

fitting at a predefined frequency, eliminates the first derivatives of the phase-lag function at the same frequency. The result is that the phase-lag function becomes less sensitive on the frequency near the predefined one. This behavior compensates the fact that the exact frequency can only be estimated. Experimental results demonstrate this behavior by showing that the accuracy is increased as the number of derivatives that are eliminated is increased.

More specifically, the new methods have been tested to a real problem known as the five-outer planet problem and the test problem of two bodies. They seem to always perform better than the corresponding classical method of Quinlan–Tremaine, and more better with low values of eccentricity. The new property allows the methods to perform well during the integration of many orbital problems and nonstiff oscillatory problems.

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