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# Evolution of Correlations in Complex Networks

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**Abstract.** We investigate in detail the mechanisms under which degree correlations evolve in complex networks. We consider the case where a vertex is entering the network at each time, carrying a predefined number of edges. We prove in this work, that the same elementary interactions which is responsible for emerging of scaling in complex networks, can give several patterns of degree correlations. As a test case, the effect of preferential attachment rule in degree correlations is studied in detail.

## 1. Introduction

During the past years there has been a vast progress in the field of complex networks [1], [2]. Various networks in nature, science and technology display, so called, complex phenomena. Networks such as the Internet, at the autonomous level or even the World Wide Web, the protein correlations network or social networks such as the friendship network or the sexual network [3], [4].

Strikingly, many of these networks have complex topological properties and dynamical features that cannot be accounted for by classical graph modelling [5]. For example, small-world properties [6] and scale-free (SF) degree distributions [7] (where the degree of a node is defined as the number of other nodes to which it is attached) seem to influence significantly the topology of real world networks. These global properties show a large connectivity and a short average distance between nodes, which have considerable impact on the behaviour of physical processes taking place on the network.

Interactions in networks may be described in terms of graphs, consisting of vertices and edges, where they represent the connections in the graph (their interactions). This approach was initiated by Erds and Rnyi (ER) [5]. In the ER model, the number of vertices is fixed, while edges connecting one vertex to another occur randomly with certain probability. However, the ER model is too random to describe real complex systems. Earlier studies concentrated on simple random networks, and it was recently discovered that many complex networks are hierarchically organised. Recently, Watts and Strogatz (WS) [6] introduced a small-world network, where a fraction of edges on a regular lattice is rewired with probability  $p_{WS}$  to other vertices. More recently, Barabasi and Albert (BA) [7], [9], [10] and [11], introduced a new approach of an evolving network where the number of vertices  $N$  increases linearly with time (rather than fixed), and a newly introduced vertex connects to the preexisting network with  $m$  edges, following the so-called preferential attachment (PA) rule. The PA rule means that the probability for the new

vertex to connect to an already existing vertex is proportional to the degree  $k$  of the selected vertex. Then the degree distribution  $P_D(k)$  follows a power law  $P_D(k) \sim k^{-\gamma}$  with  $\gamma = 3$  for the BA model, while for the ER and WS models, it follows a Poisson distribution. In SF networks, one may wonder if the exponent  $\gamma$  is universal in analogy with the theory of critical phenomena. However, the exponent  $\gamma$  turns out to be sensitive to the detail of network structure. Thus, a universal quantity for SF networks is yet to be found.

In this work, we derive a master equation for the degree correlation of an evolving network. The derivation is presented in section 2, while in section 3, simulating results are produced in the case of preferential attachment rule. Finally, in section 4, conclusions are drawn and open questions for future work are posed.

## 2. Evolution of Correlations

Suppose that  $p_k$  is the degree distribution of our network, i.e., the fraction of vertices in the network with degree  $k$ , or equivalently the probability that a vertex chosen uniformly at random from the network will have degree  $k$ . The vertex at the end of a randomly chosen edge in the network will have degree distributed in proportion to  $kp_k$ , the extra factor of  $k$  arising because  $k$  counts the available edges of a node with degree  $k$  [14], [15] and [16]. In the absence of correlations, the probability that a randomly selected edge has endpoints with degrees  $(k, k')$  is proportional to  $k \cdot p_k \cdot k' \cdot p_{k'}$ . On the other hand, degree correlations dictate that there is a conditional degree distribution  $P(k'|k)$  which is the probability that a vertex with degree  $k$  is connected with a vertex with degree  $k'$ .

Assume that initially there are  $m_0$  vertices with no edges in the network. At each time  $t > m_0$ , a node is entering the network and makes  $m$  connections with other vertices already in the network (suppose that  $m \leq m_0$ ). We define as the  $k$ -island the set of vertices that have degree  $k$  and let  $E_{k,k'}^t$  be the number of edges connecting the  $k$  and  $k'$  islands (that is the number of edges that their endpoints have degrees  $(k, k')$ ). Let  $N_k^t$  be the number of vertices in the  $k$ -island at time  $t$ ,  $P^t(k'|k)$  the conditional degree distribution at time  $t$ ,  $\langle k \rangle^t$  the mean degree at time  $t$ ,  $N^t$  the number of nodes in the network at time  $t$  (which is equal to  $t$ ),  $P^t(k)$  the degree distribution at time  $t$ ,  $P^t(k, k')$  the joint degree distribution at time  $t$  which is given by

$$P^t(k, k') = \frac{E_{k,k'}^t}{\langle k \rangle^t N^t} \quad (1)$$

and  $q_k^t$  the probability that a vertex with degree  $k$  gains an edge at time  $t$ . Since at each time  $m$ -edges are created

$$\sum_{k=1}^{\infty} q_k^t N_k^t = m \quad (2)$$

The number of edges connecting the  $k$  and  $k'$  islands at time  $t + 1$  is given:

$$\begin{aligned} E_{k,k'}^{t+1} = & E_{k,k'}^t + q_{k-1}^t N_{k-1}^t P^t(k'|k-1)(k-1) \\ & + q_{k'-1}^t N_{k'-1}^t P^t(k|k'-1)(k'-1) \\ & - q_k^t N_k^t P^t(k'|k)k \\ & - q_{k'}^t N_{k'}^t P^t(k|k')k' \\ & + q_{k'-1}^t N_{k'-1}^t \delta_{k,m} \\ & + q_{k-1}^t N_{k-1}^t \delta_{k',m} \end{aligned} \quad (3)$$

where  $\delta$  is the Kronecker symbol. The first term in equation 3 says that a new vertex connects to another vertex with degree  $k - 1$  with probability  $q_{k-1}^t N_{k-1}^t$  since we have  $N_{k-1}^t$  vertices with degree  $k - 1$ . This vertex has  $P^t(k'|k-1)(k-1)$  links with vertices with degree  $k$ . Since the connected vertex with degree  $k - 1$  has now degree  $k$ , those links must be counted now to  $E_{k,k'}^{t+1}$

. The second term in equation 3 has the same meaning (interchange  $k$  and  $k'$ ). The third term says that a new vertex connects to another vertex with degree  $k$  with probability  $q_k^t N_k^t$ . This vertex has  $P^t(k'|k)k$  links with vertices with degree  $k'$ . Since the connected vertex with degree  $k$  will have now degree  $k + 1$ , these links are not counted now to  $E_{k,k'}^t$ . The fourth term has the same meaning. Finally, the fifth and sixth term counts the new edges that connect the  $m$ -island with the other islands (note here that the new vertex enters the  $m$ -island). It can be easily verified that

$$\sum_{k,k'>0} E_{k,k'}^{t+1} = \sum_{k,k'>0} E_{k,k'}^t + 2m \tag{4}$$

as expected. For the initial condition at time  $t = m_0$  we have:

$$N_k^{m_0} = m_0 \delta_{k,0}, \quad P^{m_0}(k) = \delta_{k,0}, \quad \text{and } P^{m_0}(k, k') = 0 \tag{5}$$

Equation 3 along with the initial conditions 5 can produce  $E_{k,k'}$  at any time given  $q_k^t$ . Since

$$P(k, k') = \frac{E_{k,k'}}{\langle k \rangle N} \tag{6}$$

and

$$kP(k)P(k'|k) = \langle k \rangle P(k, k') \tag{7}$$

we can divide 3 with  $\langle k \rangle^{t+1} N^{t+1} = \langle k \rangle^{t+1} (t + 1)$ . The result, after some rearrangements, reads :

$$t \left( \frac{\langle k \rangle^{t+1}}{\langle k \rangle^t} P^{t+1}(k, k') - P^t(k, k') \right) + \frac{\langle k \rangle^{t+1}}{\langle k \rangle^t} P^{t+1}(k, k') + (Q^t(k) + Q^t(k')) P^t(k, k') = \\ Q^t(k - 1)P^t(k - 1, k') + Q^t(k' - 1)P^t(k, k' - 1) + \\ Q^t(k - 1) \frac{P^t(k - 1)}{\langle k \rangle^t} \delta_{k',m} + Q^t(k' - 1) \frac{P^t(k' - 1)}{\langle k \rangle^t} \delta_{k,m} \tag{8}$$

where  $Q^t(k) = tq_k^t$ .

Taking now the limiting case where  $t \rightarrow \infty$ , we have  $\langle k \rangle^{t+1} = \langle k \rangle^t$  and  $P^{t+1}(k, k') = P^t(k, k')$  thus

$$P(k, k') = \frac{Q(k - 1)P(k - 1, k') + Q(k' - 1)P(k, k' - 1)}{1 + Q(k) + Q(k')} + \\ \frac{Q(k - 1)P(k - 1)\delta_{k',m} + Q(k' - 1)P(k' - 1)\delta_{k,m}}{2m(1 + Q(k) + Q(k'))} \tag{9}$$

where  $Q(k) \stackrel{t \rightarrow \infty}{=} Q^t(k)$ . Since

$$P(k) = \frac{\langle k \rangle}{k} \sum_{k'>0} P(k, k') \tag{10}$$

and summing equation 9 for  $k' > 0$  we have

$$P(k) = \frac{Q(k - 1)P(k - 1)}{1 + Q(k)}, \quad k > m \tag{11}$$

and

$$P(m) = \frac{1}{1 + Q(m)} \tag{12}$$

Equation 11 can be rewritten as

$$P(v) + Q(v)P(v) = Q(v - 1)P(v - 1) \tag{13}$$

Summing for  $v = m + 1$  until  $k$  we have

$$F(k) - 1 + Q(k)P(k) = 0 \tag{14}$$

where  $F(k)$  is the cumulative degree distribution and we have used equation 12. In the case of scale free networks, for large  $k$  we must have

$$F(k) = 1 - Ck^{1-\gamma} \tag{15}$$

where after substitution in 14 and taking into account that  $P(k) = F'(k)$  we have

$$Q(k) \stackrel{k \gg 1}{\approx} \frac{k}{\gamma - 1} \tag{16}$$

### 3. Simulation Results

As an example, we examine the evolution of correlations in the well studied Barabasi-Albert network [7], In this network, the probability of a vertex with degree  $k$  to gain an edge at time  $t$  is given:

$$q_k^t = m \frac{k}{\sum_{j=1}^{N^t} k_j} \tag{17}$$

where the sum in the denominator is running on all network vertices and thus

$$q_k^t = m \frac{k}{2mt} \tag{18}$$

since the sum of all degrees is twice the number of edges and  $N^t = t$ . For  $Q(k)$  we have

$$Q(k) = \frac{k}{2} \tag{19}$$

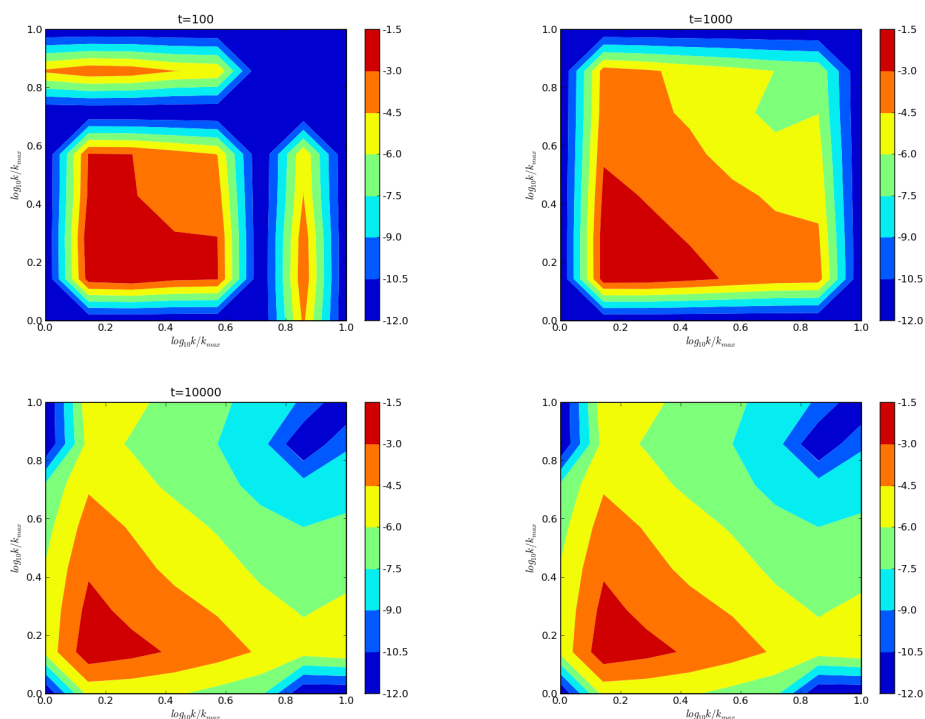
and thus the produced network is scale free with exponent equal to 3. Equation 8 now becomes

$$\begin{aligned} t \left( P^{t+1}(k, k') - P^t(k, k') \right) + P^{t+1}(k, k') + \frac{k + k'}{2} P^t(k, k') = \\ \frac{k - 1}{2} P^t(k - 1, k') + \frac{k' - 1}{2} P^t(k, k' - 1) + \\ \frac{k - 1}{2} \frac{P^t(k - 1)}{m} \delta_{k',m} + \frac{k' - 1}{2} \frac{P^t(k' - 1)}{m} \delta_{k,m} \end{aligned} \tag{20}$$

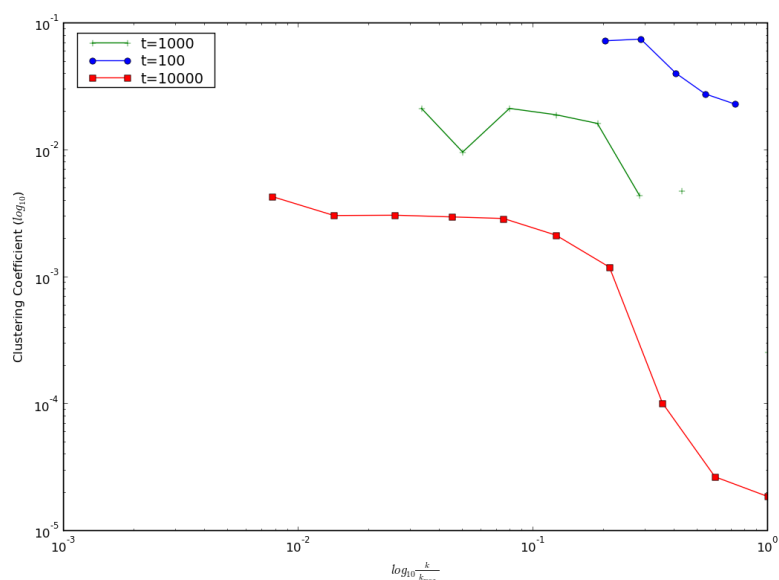
with initial conditions

$$\begin{aligned} P^{m_0}(k, k') &= 0 \\ P^{m_0}(k) &= \delta_{k,0} \end{aligned}$$

Figure 1 shows the calculated joint degree distribution for  $t = 10^2, 10^3, 10^4$  and  $\infty$ . The disassortative nature of the Barabasi-Albert network is obvious for large times, while this is not the case when the network is small enough. This happens because initially, the edges are quite uniformly distributed among the vertices, but as the time increases, some vertices collect the majority of new edges (emergence of scaling). These vertices, have small probability that they are connected due to the sparse nature of the network. On the other hand, as time increases, since neighbors of vertices with high degree have small degree, it is expected that the clustering coefficient will decrease with increasing node degree. This is shown in figure 2 where the clustering coefficient is calculated for  $t = 10^2, 10^3, 10^4$ .



**Figure 1.** Calculated joint degree distribution for  $t = 10^2, 10^3, 10^4$  and  $\infty$  for an evolving network following the preferential attachment rule.



**Figure 2.** Calculated distribution of clustering coefficient for  $t = 10^2, 10^3, 10^4$  for an evolving network following the preferential attachment rule.

#### 4. Conclusions

It has been shown in this work, that the same elementary processes that are responsible for the scale free structure of real world complex networks, can produce different patterns of degree correlations. This result comes from the construction of a master equation for the evolution of degree correlations in an evolving network in which the only parameter is the probability that a vertex with a predefined degree gains an edge in a specific time ( $Q^t(k)$ ). The master equation was tested in the Barabasi-Albert network where the well studied behaviour of these networks was reproduced. The master equation for the evolution of correlations can be used to model various kinds of networks by simply changing the probability  $Q^t(k)$ .

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